1st-Year Mathematics: Complex Analysis

Problem Set 1 November 29, 2010

1. The Euler equation is $\cos\theta + i\,\sin\theta = e^{i\,\theta}$. By making an appropriate choice for θ, use this equation to derive the following trigonometric identities:

$$\cos\bigl(\phi + \tfrac{1}{2}\pi\bigr) = -\sin\phi\,,$$
$$\sin\bigl(\phi + \tfrac{1}{2}\pi\bigr) = \cos\phi\,.$$

2. Given that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, derive identities for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

3. If $z_1 = 2 + 2\,i$ and $z_2 = -1 + 3\,i$, find

   (a) z_1^{10} (b) z_2^{-4} (c) $(z_1^*)^{10}$

   Express your answers in the form $x + i\,y$.

4. Find the real and imaginary parts of

   (a) e^z (b) e^{z^*} (c) e^{3z} (d) e^{z^2} (e) e^{iz}

5. Two complex numbers $z = x + i\,y$ and $z' = x' + i\,y'$ are equal if any only if their real and imaginary parts are equal, i.e. if and only if $x = x'$ and y'. Use this to show that, in the polar representation, $z = r\,e^{i\theta}$ and $z' = r'e^{i\theta'}$, z and z' are equal if and only if $r = r'$ and $\theta = \theta'$.

6. Show that, for any two complex numbers $z = x + i\,y$ and $z' = x' + i\,y'$,

   (a) $|z\,z'| = |z||z'|\,.$

   (b) $\left|\dfrac{z}{z'}\right| = \dfrac{|z|}{|z'|}\,.$

   Hint: Use the polar representation of complex numbers.

7. Show that $|e^{i\theta}| = 1$. Hence, show that, if $z = x + i\,y$, $|e^z| = |e^x|$.

8. Show that $(e^z)^* = e^{z^*}$.

9. Find all values of $z^8 = i$, i.e. the 8th roots of i, and plot them in the complex plane.

1st-Year Mathematics: Complex Analysis

Problem Set 2 December 7, 2010

Problems marked with an asterisk (∗) are optional.

1. We have seen that there are similarities and differences between e^x and e^z. There follow some additional examples. In each case either prove your answer or provide an example where the statement is false.

   (a) The function e^x is increasing, i.e. $e^{x_1} < e^{x_2}$ if $x_1 < x_2$. If $|z_1| < |z_2|$, is $|e^{z_1}| < |e^{z_2}|$?

   (b) The function e^x never vanishes. Can e^z vanish?

   (c) $e^x = 1$ if and only if $x = 0$. Do we have that $e^z = 1$ if and only if $z = 0$?

2. Express the following functions in the form of $u(x, y) + i\,v(x, y)$, where u and v are the real and imaginary parts of these functions.

   (a) $\sin(2z)$ (b) $\cos(z^2)$ (c) $2z + \sin z$ (d) $z \cos z$

3. Explain why the absolute values of $\cos z$ and $\sin z$ are not bounded.

4. Evaluate the following logarithms

   (a) $\ln(2\,i)$ (b) $\ln(-3 - 3\,i)$ (c) $\ln(4\,e^{\frac{1}{4}i\pi})$

5. Evaluate the following powers:

   (a) 5^i (b) $(1 + i)^{3+i}$ (c) $(-5)^{1-i}$ (d) $\left(\dfrac{1+i}{1-i}\right)^i$

7.* One justification of why $0! = 1$ is based on first defining a function f such that

$$f(n) = n! = n(n-1)(n-2)\cdots 1\,.$$

Then, by writing

$$f(n) = nf(n-1) = n(n-1)f(n-2) = \cdots = n(n-1)(n-2)\cdots 1f(0) = n!f(0)\,,$$

and using the definition $f(n) = n!$, we deduce that $f(0) = 1$. Can we find such a function f?

Consider the complex function, which is known as the **gamma function**:

$$\Gamma(z) \equiv \int_0^\infty e^{-t}\,t^{z-1}\,dt\,,$$

in which $\mathrm{Re}(z) > 0$. This one of the most common examples of a non-elementary function, that is, a function that cannot be expressed in finite terms with algebraic operations defined in terms of powers, exponentials, and logarithms.

(a) Use integration by parts to prove the following property:

$$\Gamma(z+1) = z\Gamma(z)\,.$$

(b) Show by direct computation that $\Gamma(1) = 1$.

(c) Use (a) and (b) to show that, for any positive integer n, the gamma function satisfies the recurrence relation

$$\Gamma(n) = (n-1)!\,.$$

This is why the gamma function is referred to as the **generalized factorial function**.

(d) Discuss the recurrence relation in (c) with regard to the value of 0!.

(e) Evaluate $\Gamma(\frac{1}{2})$. Begin by substituting the value into the gamma function:

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt\,.$$

Use the substitution $t = s^2$ to transform this integral to

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-s^2}\,ds = \int_{-\infty}^\infty e^{-s^2}\,ds\,.$$

The integral on the right-hand side of this equation has the value $\sqrt{\pi}$, so

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}\,.$$

(f) Use the results of (a) and (e) to calculate $\Gamma(-\frac{1}{2})$.

8.* **Hyperbolic transformations** can be constructed in an analogous manner to ordinary rotations. Consider the position 'vector' $\boldsymbol{v}$ on the unit hyperbola at the 'angle' t:

$$\boldsymbol{v} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

Apply the hyperbolic rotation $\mathsf{R}(u)$ by a hyperbolic angle u,

$$\mathsf{R}(u) = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix},$$

to the hyperbolic vector and show that

$$\boldsymbol{v}' = \mathsf{R}(u)\boldsymbol{v} = \begin{pmatrix} \cosh(t+u) \\ \sinh(t+u) \end{pmatrix},$$

which is another point on the hyperbola.

1st-Year Mathematics: Complex Analysis & Differential Equations

Problem Set 3 December 14, 2010

1. **Newton's law of cooling** states that the temperature of a body changes at a rate proportional to the difference in temperature between the body and that of its environment. The differential equation for the temperature $T(t)$ of a body at time t in an environment whose ambient temperature is θ is

$$\frac{dT}{dt} = -k(T - \theta)\,,$$

where k is a positive constant. Show that the solution of this equation with the initial condition $T(0) = T_0$ is

$$T(t) = \theta + (T_0 - \theta)\,\mathrm{e}^{-kt}\,.$$

Hint: Begin by obtaining the differential equation for $u(t) = T(t) - \theta$, using the fact that θ is a constant. Note that the corresponding initial condition is $u(0) = T_0 - \theta$.

2. Consider the equation of motion of a classical undamped harmonic oscillator with natural frequency ω_0,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,,$$

with the initial conditions

$$x(0) = x_0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = x_0'\,.$$

Show that the solution to this initial value problem is

$$x(t) = x_0 \cos(\omega_0 t) + \frac{x_0'}{\omega_0}\sin(\omega_0 t)\,.$$

3. The differential equation

$$\alpha\frac{d^2u}{dx^2} + \beta\frac{d}{dx}(xu) = \frac{d}{dx}\left(\alpha\frac{du}{dx} + \beta xu\right) = 0\,,$$

in which α and β are constants, arises in certain solutions of the **Fokker–Planck equation**, which describes the equilibration of macroscopic systems.

(a) Show that this equation implies that

$$\alpha\frac{du}{dx} + \beta xu = A\,.$$

where A is a constant.

(b) Suppose that we are interested in the solution whose behavior at large values of x is given by

$$\lim_{x\to\infty}\big[xu(x)\big]=0 \qquad \text{and} \qquad \lim_{x\to\infty}\left(\frac{du}{dx}\right)=0\,.$$

Show that these conditions imply that the constant A must vanish, so the equation reduces to

$$\alpha\frac{du}{dx}+\beta xu=0\,.$$

(c) The equation obtained in (b) is a separable first-order differential equation. Show that the solution to this equation is

$$u(x)=B\,\mathrm{e}^{-\beta x^2/2\alpha}\,,$$

where B is a constant. With an appropriate choice of α and β, this solution is the Maxwell–Boltzmann distribution.

4. According to the theory of beam bending, the deflection y of a beam from the horizontal is determined by a fourth-order differential equation of the form

$$\frac{d^4y}{dx^4}-y=0\,.$$

Determine the general solution of this equation. How many auxiliary conditions are required to determine the arbitrary constants in the general solution?

5. The Schrödinger equation that describes the stationary states of a particle of mass m in a one-dimensional potential $V(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

where $\hbar=h/2\pi$, E is the energy of the particle and ψ is the wavefunction. Determine the general solution to this equation for the case where the potential is independent of position, i.e. where $V(x)=V$ is a constant. How does the relation between E and V determine the character of the solutions? How many auxiliary conditions are required to determine the arbitrary constants in this solution?

1st-Year Mathematics: Complex Analysis

Solutions to Problem Set 1 November 29, 2010

1. Substitute $\phi + \frac{1}{2}\pi$ into the Euler equation $\cos\theta + i\,\sin\theta = e^{i\,\theta}$ to obtain

$$\begin{aligned}\cos\big(\phi + \tfrac{1}{2}\pi\big) + i\,\sin\big(\phi + \tfrac{1}{2}\pi\big) &= e^{i\,(\phi+\frac{1}{2}\pi)}\\ &= e^{\frac{1}{2}\pi\,i}\,e^{i\phi}\\ &= i\,(\cos\phi + i\,\sin\phi)\\ &= -\sin\phi + i\cos\phi\,.\end{aligned}$$

   Equating real and imaginary parts yields

$$\cos\big(\phi + \tfrac{1}{2}\pi\big) = -\sin\phi\,, \qquad \sin\big(\phi + \tfrac{1}{2}\pi\big) = \cos\phi\,.$$

2. De Moivre's formula with $n = 3$ is

$$(\cos\theta + i\,\sin\theta)^3 = \cos 3\theta + i\,\sin 3\theta\,.$$

   Given that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\,,$$

   the expansion of left-hand side of yields

$$\cos^3\theta + 3i\,\cos^2\theta\sin\theta - 3\cos\theta\sin\theta - i\sin^3\theta = \cos 3\theta + i\,\sin 3\theta\,.$$

   By equating real and imaginary parts, we obtain

$$\begin{aligned}\cos 3\theta &= \cos^2\theta - 3\sin\theta\cos\theta\,,\\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^2\theta\,.\end{aligned}$$

3. Given that $z_1 = 2 + 2\,i$ and $z_2 = -1 + 3\,i$, the corresponding polar representations are

$$z_1 = 2\sqrt{2}\,e^{i\theta}\,, \qquad \cos\theta = \frac{\sqrt{2}}{2}\,, \qquad \sin\theta = \frac{\sqrt{2}}{2}\,,$$

   so $\theta = \frac{1}{4}\pi$, and

$$z_2 = \sqrt{10}\,e^{i\theta}\,, \qquad \cos\theta = -\frac{\sqrt{10}}{10}\,, \qquad \sin\theta = \frac{3\sqrt{10}}{10}\,.$$

(a) This calculation is most easily carried out in the polar representation:

$$z_1 = 2\sqrt{2}\, e^{i\theta}\,, \qquad \cos\theta = \frac{\sqrt{2}}{2}\,, \qquad \sin\theta = \frac{\sqrt{2}}{2}\,,$$

so $\theta = \frac{1}{4}\pi$. Hence,

$$z_1^{10} = (2\sqrt{2})^{10}\, e^{10i\theta} = 8^5\, e^{\frac{10}{4}\pi i} = 8^5\, e^{\frac{1}{2}i\pi} = 8^5\, i\,.$$

(b) This calculation is most easily carried out in the rectangular representation. We first determine z_2^4:

$$\begin{aligned} z_2^4 &= (z_2^2)^2 = \big[(-1+3\,i)^2\big]^2 = (1-6\,i-9)^2 \\ &= (-8-6\,i)^2 = 64+96\,i-36 = 28+96\,i\,. \end{aligned}$$

(c) Using the polar representation $z = r\,e^{i\theta}$,

$$(z^n)^* = \Big[\big(r\,e^{i\theta}\big)^n\Big]^* = \big(r^n\,e^{in\theta}\big)^* = r^n\,e^{-in\theta} = \big(r\,e^{-i\theta}\big)^n = (z^*)^n\,.$$

Hence,

$$(z_1^*)^{10} = (z_1^{10})^* = -2^5\,i\,.$$

4. (a)

$$e^z = e^{x+i\,y} = e^x(\cos y + i\,\sin y) = e^x\cos y + i\,e^x\sin y\,.$$

Thus,

$$\mathrm{Re}(e^z) = e^x\cos y\,, \qquad \mathrm{Im}(e^z) = e^x\sin y\,.$$

(b)

$$e^{z^*} = e^{x-i\,y} = e^x(\cos y - i\,\sin y) = e^x\cos y - i\,e^x\sin y\,.$$

Thus,

$$\mathrm{Re}(e^{z^*}) = e^x\cos y\,, \qquad \mathrm{Im}(e^{z^*}) = -e^x\sin y\,.$$

(c)

$$e^{3z} = e^{3(x+i\,y)} = e^{3x}(\cos 3y + i\,\sin 3y) = e^{3x}\cos 3y + i\,e^{x}\sin 3y\,.$$

Thus,

$$\mathrm{Re}(e^{3z}) = e^{3x}\cos 3y\,, \qquad \mathrm{Im}(e^{3z}) = e^{3x}\sin 3y\,.$$

(d)

$$\begin{aligned} e^{z^2} &= e^{(x+i\,y)^2} = e^{(x^2-y^2)+2ixy} \\ &= e^{x^2-y^2}(\cos 2xy + i\,\sin 2xy) = e^{x^2-y^2}\cos 2xy + i\,e^{x^2-y^2}\sin 2xy\,. \end{aligned}$$

Thus,

$$\mathrm{Re}(e^{z^2}) = e^{x^2-y^2}\cos 2xy\,, \qquad \mathrm{Im}(e^{z^2}) = e^{x^2-y^2}\sin 2xy\,.$$

(e)

$$e^{iz} = e^{i(x+i\,y)} = e^{-y+ix} = e^{-y}(\cos x + i\,\sin x) = e^{-y}\cos x + i\,e^{-y}\sin x\,.$$

Thus,

$$\mathrm{Re}(e^{iz}) = e^{-y}\cos x\,, \qquad \mathrm{Im}(e^{iz}) = e^{-y}\sin x\,.$$

5. Using the polar representation for

$$z = x + iy = r\,e^{i\theta}\,, \qquad z' = x' + iy' = r'\,e^{i\theta'}\,,$$

we have that $z = z'$ implies that $r\,e^{i\theta} = r'\,e^{i\theta'}$. Since $x = x'$ and $y = y'$,

$$r' = \sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2} = r\,,$$

and

$$\cos\theta' = \frac{x'}{r'} = \frac{x}{r} = \cos\theta\,, \qquad \sin\theta' = \frac{y'}{r'} = \frac{y}{r} = \sin\theta\,,$$

from which we conclude that $\theta = \theta'$. We are, of course, restricting ourselves to the range $0 \le \theta < 2\pi$.

6. Using the polar representation for

$$z = x + iy = r\,e^{i\theta}\,, \qquad z' = x' + iy' = r'\,e^{i\theta'}\,,$$

we have

(a)

$$|z\,z'| = |r\,e^{i\theta} r'\,e^{i\theta'}| = |r\,r'\,e^{i(\theta+\theta')}| = r\,r' = |z|\,|z'|\,,$$

(b)

$$\left|\frac{z}{z'}\right| = \left|\frac{r}{r'}\,e^{i(\theta-\theta')}\right| = \frac{r}{r'} = \frac{|z|}{|z'|}\,.$$

7. With $e^{i\theta} = \cos\theta + i\,\sin\theta$, we have

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1\,.$$

Hence,

$$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = |e^x|\,.$$

8. For $z = x + i\,y$, we have that

$$\begin{aligned}(e^z)^* &= (e^{x+iy})^* = \big[e^x(\cos y + i\,\sin y)\big]^* \\ &= e^x(\cos y - i\,\sin y) = e^x\,e^{-iy} = e^{x-iy} = e^{z^*}\,.\end{aligned}$$

9. The 8th roots of i satisfy the equation $z^8 = i$. In polar form, $z = r\,e^{i\theta}$ and $i = e^{\frac{1}{2}\pi i}$, in which case our equation becomes

$$r\,e^{8i\theta} = e^{\frac{1}{2}\pi i}\,.$$

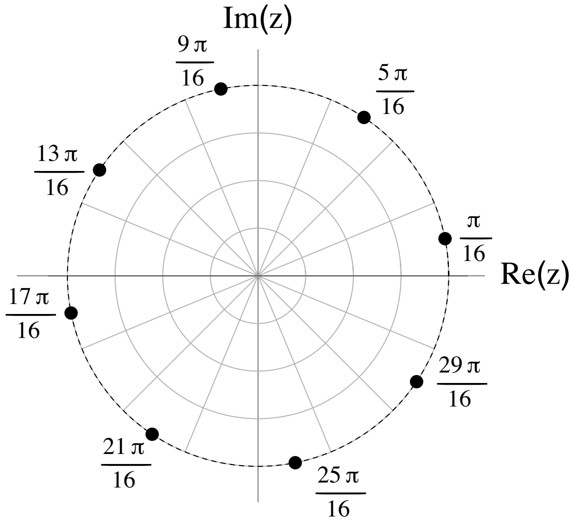
We must therefore have $r = 1$ and

$$8\theta = \tfrac{1}{2}\pi,\ \tfrac{1}{2}\pi + 2\pi, \ldots, \tfrac{1}{2}\pi + 14\pi\,,$$

or,

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}\,.$$

These are shown below in the complex plane:



10. From Classwork 3, the n roots of unity z_n can be written as

$$z_n = \exp\left(\frac{2k\pi i}{n}\right).$$

for $k = 0, 1, 2, \dots, n-1$. With $z_0 = 1$, we see that by defining

$$\omega = \exp\left(\frac{2\pi i}{n}\right),$$

we have

$$\omega^k = \underbrace{\exp\left(\frac{2\pi i}{n}\right) \cdots \exp\left(\frac{2\pi i}{n}\right)}_{n \text{ factors}} = \exp\left(\frac{2k\pi i}{n}\right).$$

Thus, the nth roots of unity can bed written as $1, \omega, \omega^2, \dots, \omega^{n-1}$.

1st-Year Mathematics: Complex Analysis

Solutions to Problem Set 2 December 14, 2010

1. (a) We have that

$$\left|e^z\right| = \left|e^{x+i\,y}\right| = \left|e^x\right|\left|e^{iy}\right| = e^x\,,$$

because e^x is a positive real number. Since $|z| = (x^2+y^2)^{1/2}$, and the magnitude of e^z depends only on the real part of z, there is not a monotonic relationship between $|z|$ and $|e^z|$.

(b) Given that

$$e^z = e^{x+i\,y} = e^x\,e^{iy} = e^x(\cos y + i\,\sin y)\,,$$

we see that there is no value of y for which both $siny$ and $\cos y$ vanish.

(c) As in (b),

$$e^z = e^x(\cos y + i\,\sin y)\,.$$

Thus, $e^z = 0$ is $x = 0$ and if $y = 0, 2\pi, 4\pi\ \ldots.$ Only if we restrict the range of y to $0 \le y < 2\pi$ is this statement true.

2. We use the following decompositions for this problem:

$$\begin{aligned}\sin(u+iv) &= \frac{1}{2i}\left(e^{i(u+iv)} - e^{-i(u+iv)}\right)\\ &= \frac{1}{2i}\left(e^{-v}e^{iu} - e^{v}e^{-iu}\right)\\ &= \frac{1}{2i}\left[e^{-v}(\cos u + i\,\sin u) - e^{v}(\cos u - i\,\sin u)\right]\\ &= \frac{1}{2i}\left[\left(e^{-v} - e^{v}\right)\cos u + i\left(e^{-v} + e^{v}\right)\sin u\right]\\ &= \sin u\cosh v + i\,\cos u\sinh v\,, \qquad (1)\end{aligned}$$

where $u = u(x,y)$ and $v = v(x,y)$, and similarly,

$$\begin{aligned}\cos(u+iv) &= \frac{1}{2}\left(e^{i(u+iv)} + e^{-i(u+iv)}\right)\\ &= \frac{1}{2}\left(e^{-v}e^{iu} + e^{v}e^{-iu}\right)\\ &= \frac{1}{2}\left[e^{-v}(\cos u + i\,\sin u) + e^{v}(\cos u - i\,\sin u)\right]\\ &= \frac{1}{2}\left[\left(e^{-v} + e^{v}\right)\cos u + i\left(e^{-v} - e^{v}\right)\sin u\right]\\ &= \cos u\cosh v - i\,\sin u\sinh v\,, \qquad (2)\end{aligned}$$

(a) We use (1) with $u + iv = 2z = 2x + 2iy$, so $u = 2x$ and $v = 2y$:

$$\begin{aligned}\operatorname{Re}(\sin 2z) &= \sin 2x \cosh 2y\,,\\ \operatorname{Im}(\sin 2z) &= \cos 2x \sinh 2y\,.\end{aligned}$$

(b) We use (2) with $u + iv = z^2 = x^2 - y^2 + 2ixy$, so $u = x^2 - y^2$ and $v = 2xy$:

$$\begin{aligned}\operatorname{Re}(\cos z^2) &= \cos(x^2 - y^2)\cosh(2xy)\,,\\ \operatorname{Im}(\cos z^2) &= -\sin(x^2 - y^2)\sinh(2xy)\,.\end{aligned}$$

(c) We use (1) with $u + iv = z = x + iy$, so $u = x$ and $v = y$:

$$\begin{aligned}\operatorname{Re}(2z + \sin z) &= 2\operatorname{Re}(z) + \operatorname{Re}(\sin z) = 2x + \sin x \cosh y\,,\\ \operatorname{Im}(2z + \sin z) &= 2\operatorname{Im}(z) + \operatorname{Im}(\sin z) = 2y + \cos x \sinh y\,.\end{aligned}$$

(d) We use (2) with $u + iv = z = x + i\,y$, so $u = x$ and $v = y$. Thus,

$$\begin{aligned}z \cos z &= (x + i\,y)(\cos x \cosh y - i\,\sin x \sinh y)\\ &= x \cos x \cosh y + y \sin x \sinh y + i(y \cos x \cosh y - x \sin x \sinh y)\,.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{Re}(z \cos z) &= x \cos x \cosh y + y \sin x \sinh y\,,\\ \operatorname{Im}(z \cos z) &= y \cos x \cosh y - x \sin x \sinh y\,.\end{aligned}$$

3. Consider first the complex cosine function. From (1),

$$\cos z = \cos x \cosh y - i\,\sin x \sinh y\,.$$

Thus,

$$\begin{aligned}|\cos z| &= \left(\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y\right)^{1/2}\\ &= \left[\cos^2 x \cosh^2 y + (1 - \cos^2 x)\sinh^2 y\right]^{1/2}\\ &= \left[\cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y\right]^{1/2}\\ &= \left(\cos^2 x + \sinh^2 y\right)^{1/2}.\end{aligned}$$

Since

$$\sinh 2y = \frac{e^{2y} - e^{-2y}}{2} \,,$$

we see that

$$\lim_{y\to\infty} |\cos z| \to \tfrac{1}{2}e^{y} \to \infty \,.$$

An analogous argument shows that the complex sine function is also unbounded.

4. (a)

$$\ln(-1) = \ln e^{i\pi + 2n\pi i} = i\pi + 2n\pi \,,$$

for any integer n. The principal value corresponds to $n = 0$.

(b)

$$\ln i = \ln e^{\frac{1}{2}i\pi + 2n\pi i} = \tfrac{1}{2}i\pi + 2n\pi \,,$$

for any integer n. The principal value corresponds to $n = 0$.

5. (a) With $2i = 2\,e^{\frac{1}{2}i\pi}$, we have

$$\ln(2i) = \ln\bigl(2\,e^{\frac{1}{2}i\pi}\bigr) = \ln 2 + \tfrac{1}{2}i\pi \,.$$

(b) With $-3 - 3\,i = 3\sqrt{2}\,e^{i\theta}$, where

$$\cos\theta = \sin\theta = -\frac{3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2} \,,$$

so $\theta = \frac{5}{4}\pi$, we have

$$\ln(-3 - 3\,i) = \ln\bigl(3\sqrt{2}e^{i\theta}\bigr) = \ln 3\sqrt{2} + \frac{5\pi i}{4} \,.$$

(c)

$$\ln(4\,e^{\frac{1}{4}i\pi}) = \ln 4 + \frac{\pi\, i}{4} \,.$$

6. (a)

$$5^i = e^{i\ln 5} = \cos\ln 5 + i\,\sin\ln 5\,.$$

(b) With $1+i = \sqrt{2}\,e^{\frac{1}{4}i\pi}$, we have

$$\begin{aligned}(1+i)^{3+i} &= e^{(3+i)\ln(1+i)} = e^{(3+i)(\ln\sqrt{2}+\frac{1}{4}i\pi} \\ &= e^{3\ln 2-\frac{1}{4}\pi}\,e^{i(\ln\sqrt{2}+\frac{3}{4}\pi)} \\ &= e^{3\ln 2-\frac{1}{4}\pi}\big[\cos(\ln\sqrt{2}+\tfrac{3}{4}\pi)+i\sin(\ln\sqrt{2}+\tfrac{3}{4}\pi)\big]\,.\end{aligned}$$

(c)

$$(-5)^{1-i} = e^{(1-i)\ln(-5)}\,.$$

With

$$\ln(-5) = \ln(5e^{i\pi}) = \ln 5 + i\pi\,,$$

we obtain

$$\begin{aligned}e^{(1-i)\ln(-5)} &= e^{(1-i)(\ln 5+i\pi)} \\ &= e^{\ln 5+i\pi}e^{i(-\ln 5+\pi)} \\ &= e^{\ln 5+i\pi}\big[\cos(-\ln 5+\pi)+i\sin(-\ln 5+\pi)\big]\,.\end{aligned}$$

(d) We begin with

$$\frac{1+i}{1-i} = \left(\frac{1+i}{1-i}\right)\left(\frac{1+i}{1+i}\right) = \frac{1+2\,i-1}{1+1} = i\,.$$

Thus,

$$i^i = e^{i\ln i} = e^{i\ln(e^{i\pi/2})} = e^{-\frac{1}{2}\pi}\,.$$

7. (a)

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt = -e^{-t}t^z\Big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}\,dt = z\Gamma(z)\,.$$

(b)

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1\,.$$

(c) Beginning with $\Gamma(1) = 1$, we have

$$\begin{aligned}\Gamma(2) &= \Gamma(1+1) = \Gamma(1) = 1\,,\\ \Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2\times 1\,,\\ \Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 3\times 2\times 1\,.\end{aligned}$$

The pattern is now clear and we can deduce that

$$\Gamma(n) = (n-1)!\,. \tag{3}$$

(d) To extend the formula (3) to all positive integers, we must have that

$$\Gamma(1) = (1-1)! = 0! = 1\,.$$

(e)

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t}t^{-1/2}\, dt = \int_0^{\int} \frac{e^{-t}}{\sqrt{t}}\, dt\,.$$

Changing the integration variable to $s^2 = t$, we have that $dt = 2s\, ds$, the limits of integration are unchanged, and the integral becomes

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-s^2}\, ds = \int_{-\infty}^\infty e^{-s^2}\, ds\,,$$

because the integrand is an even function. Calling this integral I we can evaluate it by squaring it and then transforming to polar coordinates:

$$\begin{aligned}I^2 &= \left(\int_{-\infty}^\infty e^{-x^2}\, dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\, dy\right)\\ &= \int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(x^2+y^2)}\, dx\, dy\\ &= \int_0^{2\pi} d\phi \int_0^\infty r\, dr\, e^{-r^2}\\ &= 2\pi\left(-\frac{e^{-r^2}}{2}\Big|_0^\infty\right)\\ &= \pi\,.\end{aligned}$$

Thus, $I = \sqrt{\pi}$, so

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}\,,$$

and

$$\Gamma(\tfrac{1}{2}) = -\tfrac{1}{2}\Gamma(-\tfrac{1}{2})\,,$$

so

$$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}\,.$$

The transformation of the volume element $dx\,dy$ to $r\,dr\,d\phi$ can be understood in terms of the usual transformation between rectangular and polar coordinates:

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,.$$

We now express the vector $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$ as

$$\boldsymbol{r} = r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j}\,.$$

We now calculate the vectors dr_r and dr_ϕ resulting from the differential with respect to r and ϕ, respectively:

$$\begin{aligned} dr_r &= dr\cos\phi\,i + dr\sin\phi\,j \\ dr_\phi &= -r\sin\phi\,d\phi\,i + r\cos\phi\,d\phi\,j\,. \end{aligned}$$

These vectors are orthogonal,

$$\begin{aligned} dr_r\cdot dr_\phi &= (dr\cos\phi\,i + dr\sin\phi\,j)\cdot(-r\sin\phi\,d\phi\,i + r\cos\phi\,d\phi\,j) \\ &= -r\,dr\cos\phi\sin\phi\,d\phi + r\,dr\cos\phi\sin\phi\,d\phi \\ &= 0\,, \end{aligned}$$

so the area defined by these vectors is obtained by multiplying their magnitudes:

$$dA = |dr_r|\cdot|dr_\phi| = dr\cdot r\,d\phi = r\,dr\,d\phi\,.$$

8. The effect of a hyperbolic rotation of a vector

$$\boldsymbol{v} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

by the rotation matrix $\mathsf{R}(u)$ by a hyperbolic angle u,

$$\mathsf{R}(u) = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix},$$

is determined by first performing a matrix calculation:

$$\mathsf{R}(u)\boldsymbol{v} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$$

$$= \begin{pmatrix} \cosh u \cosh t + \sinh u \sinh t \\ \sinh u \cosh t + \cosh u \sinh t \end{pmatrix}$$

By using the representation of the hyperbolic functions in terms of exponential functions,

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

we calculate

$$\begin{aligned} \cosh u \cosh t + \sinh u \sinh t &= \left(\frac{e^u + e^{-u}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{e^u - e^{-u}}{2}\right)\left(\frac{e^t - e^{-t}}{2}\right) \\ &= \frac{e^{u+t} + e^{-u+t} + e^{u-t} + e^{-u-t}}{4} + \frac{e^{u+t} - e^{-u+t} - e^{u-t} + e^{-u-t}}{4} \\ &= \frac{e^{u+t} + e^{-u-t}}{2} \\ &= \cosh(u+t), \end{aligned}$$

and

$$\begin{aligned} \sinh u \cosh t + \cosh u \sinh t &= \left(\frac{e^u - e^{-u}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{e^u + e^{-u}}{2}\right)\left(\frac{e^t - e^{-t}}{2}\right) \\ &= \frac{e^{u+t} - e^{-u+t} + e^{u-t} - e^{-u-t}}{4} + \frac{e^{u+t} + e^{-u+t} - e^{u-t} - e^{-u-t}}{4} \\ &= \frac{e^{u+t} - e^{-u-t}}{2} \\ &= \sinh(u+t), \end{aligned}$$

Hence,

$$\boldsymbol{v}' = \mathsf{R}(u)\boldsymbol{v} = \begin{pmatrix} \cosh(u+t) \\ \sinh(u+t) \end{pmatrix},$$

which is another point on the hyperbola.

1st-Year Mathematics: Complex Analysis & Differential Equations

Solutions to Problem Set 3 December 14, 2010

1. Beginning with the differential equation

$$\frac{dT}{dt} = -k(T - \theta)\,, \tag{1}$$

we introduce the function

$$u(t) = T(t) - \theta\,.$$

Since θ is a constant, the differential equation for u is

$$\frac{du}{dt} = \frac{dT}{dt} = -k(T - \theta) = -ku\,,$$

with the initial condition

$$u(0) = T(0) - \theta = T_0 - \theta\,.$$

As discussed in the lectures, the solution to this equation is

$$u(t) = A\,\mathrm{e}^{-kt}\,,$$

where A is a constant determined by the initial condition:

$$u(0) = T_0 - \theta = A\,.$$

In terms of the original function T, this solution is

$$T(t) = \theta + u(t) = \theta + (T_0 - \theta)\,\mathrm{e}^{-kt}\,. \tag{2}$$

Notice that (1) can also be integrated directly. By separating the variables in the equation,

$$\frac{dT}{T - \theta} = -kdt\,,$$

and integrating,

$$\int_{T_0}^{T(t)} \frac{dT}{T - \theta} = -k\int_0^t ds\,,$$

we obtain

$$\ln(T - \theta)\Big|_{T_0}^{T(t)} = \ln\left[\frac{T(t) - \theta}{T_0 - \theta}\right] = -kt\,.$$

Solving for $T(t)$,

$$T(t) = \theta + (T_0 - \theta)\,\mathrm{e}^{-kt}\,,$$

which is the same as (2).

2. We solve the equation of motion of a classical undamped harmonic oscillator with natural frequency ω_0,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,, \tag{3}$$

with a trial solution $x(t) = e^{mt}$. Substituting this expression into the equation yields

$$m^2\,e^{mt} + \omega_0^2\,e^{mt} = (m^2 + \omega_0^2)\,e^{mt} = 0\,.$$

The characteristic equation is

$$m^2 + \omega_0^2 = (m - i\omega_0)(m + i\omega_0) = 0\,,$$

which has roots $m_1 = -i\omega_0$ and $m_2 = i\omega_0$. The general solution to (3) is

$$x(t) = A\,e^{-i\omega_0 t} + B\,e^{i\omega_0 t}\,, \tag{4}$$

where A and B are determined by the initial conditions,

$$x(0) = x_0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = x_0'\,,$$

which correspond to an initial displacement x_0 and an initial velocity x_0'. Substitution of (4) into the initial conditions produces

$$\begin{aligned} x(0) &= A + B = x_0\,, \\ \left.\frac{dx}{dt}\right|_{t=0} &= -i\omega_0 A + i\omega_0 B = x_0'\,. \end{aligned}$$

After dividing both sides of the second equation by ω_0 and multiplying both sides by i, we obtain the two simultaneous equations for A and B in the form:

$$\begin{aligned} A + B &= x_0\,, \\ A - B &= \frac{ix_0'}{\omega_0}\,. \end{aligned}$$

These equations are easily solved to obtain

$$\begin{aligned} A &= \frac{1}{2}\left(x_0 + \frac{ix_0'}{\omega_0}\right)\,, \\ B &= \frac{1}{2}\left(x_0 - \frac{ix_0'}{\omega_0}\right)\,. \end{aligned}$$

Thus, the solution to the initial-value problem is

$$\begin{aligned} x(t) &= \frac{1}{2}\left(x_0 + \frac{ix_0'}{\omega_0}\right)\,e^{-i\omega_0 t} + \frac{1}{2}\left(x_0 - \frac{ix_0'}{\omega_0}\right)\,e^{i\omega_0 t} \\ &= x_0\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) - \frac{ix_0'}{\omega_0}\left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2}\right) \end{aligned}$$

$$= x_0 \left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right) + \frac{x_0'}{\omega_0} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right)$$

$$= x_0 \cos \omega_0 t + \frac{x_0'}{\omega_0} \sin \omega_0 t \,.$$

3. (a) The differential equation can be written as

$$\alpha \frac{d^2 u}{dx^2} + \beta \frac{d}{dx}(xu) = \frac{d}{dx}\left(\alpha \frac{du}{dx} + \beta x u \right) = 0 \,. \tag{5}$$

The second form of this equation implies that the quantity within the parentheses *must* reduce to a constant A, since its derivative with respect to x vanishes:

$$\alpha \frac{du}{dx} + \beta x u = A \,. \tag{6}$$

(b) For solutions which vanish faster than $1/x$ as $x \to \infty$ and whose first derivative vanishes in this limit as well,

$$\lim_{x\to\infty} \left(\alpha \frac{du}{dx} + \beta x u \right) = 0 \,, \tag{7}$$

so the constant A, which must be the same for *all* values of x, must be zero. Hence, the equation to be solved reduces to

$$\alpha \frac{du}{dx} + \beta x u = 0 \,. \tag{8}$$

(c) Equation (8) may be solved by separating the variables and integrating from $u_0 = u(x_0)$ to $u = u(x)$,

$$\int_{u_0}^{u(x)} \frac{du}{u} = -\frac{\beta}{\alpha} \int_{x_0}^{x} s \, ds \,. \tag{9}$$

We obtain

$$\ln \left[\frac{u(x)}{u_0} \right] = -\frac{\beta}{2\alpha}(x^2 - x_0^2) \,, \tag{10}$$

which can be rearranged as

$$u(x) = B \, \mathrm{e}^{-\beta x^2/2\alpha} \,, \tag{11}$$

where $B = u_0 \, \mathrm{e}^{\beta x_0^2/2\alpha}$ is a constant. We can verify that

$$\lim_{x\to\infty} \big[x u(x) \big] = \lim_{x\to\infty} \big(x \, \mathrm{e}^{-\beta x^2/2\alpha} \big) = 0 \tag{12}$$

and

$$\lim_{x\to\infty}\big[u_x(x)\big] = \lim_{x\to\infty}\left(-\frac{\beta x}{\alpha}\,\mathrm{e}^{-\beta x^2/2\alpha}\right) = 0\,, \tag{13}$$

since, for any m,

$$\lim_{x\to\infty}\left(x^m\,\mathrm{e}^{-x^2}\right) = 0. \tag{14}$$

4. Since this is a differential equation with constant coefficients, we attempt to find solutions with a trial solution of the form $y(x) = e^{mx}$. Substituting this expression into the differential equation yields

$$m^4\,e^{mx} - e^{mx} = (m^4 - 1)\,e^{mx} = 0\,.$$

The characteristic equation is identified as

$$m^4 - 1 = 0\,,$$

which can be factored as

$$m^4 - 1 = (m^2-1)(m^2+1) = (m-1)(m+1)(m-i)(m+i) = 0\,,$$

which yields four distinct roots: $m = -1, 1, -i, i$. Accordingly, there are four solutions of the differential equation:

$$y_1(x) = e^{-x}\,,\quad y_2(x) = e^{x}\,,\quad y_3(x) = e^{-ix}\,,\quad y_4(x) = e^{ix}\,.$$

The general solution is a general linear combination of these solutions,

$$y(x) = A\,e^{-x} + B\,e^{x} + C\,e^{-ix} + D\,e^{ix}\,,$$

where *four* initial conditions are required to determine the four constants A, B, C, and D.

5. To obtain the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \big[E - V(x)\big]y = 0\,,$$

we attempt a solution of the form $y(x) = \mathrm{e}^{mx}$ and choose m by the requirement that this expression is a solution. Substituting into the equation yields

$$(m^2 + E - V)\,\mathrm{e}^{mx} = 0\,,$$

so the characteristic equation is

$$m^2 + E - V = 0\,.$$

Thus, the values of m are

$$m_1 = \sqrt{V - E}, \qquad m_2 = -\sqrt{V - E}$$

Notice that if $V > E$, then m_1 and m_2 are *real*, while if $V < E$, the m_1 and m_2 are *imaginary* (and complex conjugates of one another). In either case, the general solution can be written as

$$\psi(x) = A\,\mathrm{e}^{m_1 x} + B\,\mathrm{e}^{m_2 x}$$

where A and B are constants to be determined by auxiliary conditions, of which there must be *two*.