1 Field and potential

1.1 Coulomb's law (1785)

Force on charge q due to charge Q

$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad [\mathrm{N}] \tag{1.1}$$

where $\hat{\mathbf{r}}$ points from Q to q, and r is the separation of the charges.

1.2 Electric field of a point charge Q

$$\mathbf{E} = \frac{\mathbf{F}}{q} \quad \text{or} \quad \mathbf{F} = q\mathbf{E} \tag{1.2}$$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad [\mathrm{N/C}] = [\mathrm{V/m}] \tag{1.3}$$

1.3 Electrostatic potential energy of a point charge

$$U = \frac{Qq}{4\pi\varepsilon_0 r} \quad [\mathrm{J}] \tag{1.4}$$

1.4 Electrostatic potential of a point charge

$$V = \frac{U}{q} \quad [\mathrm{V}] = [\mathrm{J/C}] \tag{1.5}$$

$$V = \frac{Q}{4\pi\varepsilon_0 r} \tag{1.6}$$

1.5 Links between F and U, and between E and V

$$F_r = -\frac{dU}{dr} \quad \text{or} \quad \mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}} \quad [\mathrm{N}] = [\mathrm{J/m}] \tag{1.7}$$

$$E_r = -\frac{dV}{dr} \quad \text{or} \quad \mathbf{E} = -\frac{dV}{dr}\hat{\mathbf{r}} \quad [\mathrm{V/m}] \tag{1.8}$$

$$dU = -\mathbf{F}\cdot d\mathbf{l} \quad dV = -\mathbf{E}\cdot d\mathbf{l} \tag{1.9}$$

where F_r and E_r are respectively the Coulomb force and the electric field in the direction of increasing r.
The force points in the direction of DECREASING potential energy. The electric field points in the direction of DECREASING electrostatic potential.

1.6 Potential difference

$$\Delta V \equiv V_B - V_A = -\int_{A\to B} \mathbf{E}.d\mathbf{l} \tag{1.10}$$

$$\oint_{A\to B} \mathbf{E}.d\mathbf{l} = 0 \tag{1.11}$$

The integral in Eq. (1.10) is known as a line (or path) integral. When the path is closed, as in Eq. (1.11), it is known as a loop integral.

1.7 Relationship between E and V in Cartesian co-ordinates and vector notation

$$E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z}. \tag{1.12}$$

$$\mathbf{E} = -\left(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}\right) = -\nabla V \tag{1.13}$$

1.8 Conservative fields

The following are equivalent signatures of a conservative field:

- $\int \mathbf{E} \cdot d\mathbf{l}$ is independent of path
- the loop integral is zero (Eq. (1.11));
- $\mathbf{E} = -\nabla V$ (Eq. (1.13), or the component relations Eq. (1.12)).

1.9 Principle of superposition

Force on q due to Q_1 and Q_2 :

$$\mathbf{F} = \frac{q}{4\pi\varepsilon_0}\left(\frac{Q_1\hat{\mathbf{r}}_1}{r_1^2} + \frac{Q_2\hat{\mathbf{r}}_2}{r_2^2}\right) = \mathbf{F}_1 + \mathbf{F}_2 \tag{1.14}$$

Electric field of Q_1 and Q_2 at given location

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1\hat{\mathbf{r}}_1}{r_1^2} + \frac{Q_2\hat{\mathbf{r}}_2}{r_2^2}\right) = \mathbf{E}_1 + \mathbf{E}_2 \tag{1.15}$$

Electric potential of Q_1 and Q_2 at given location

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2}\right) = V_1 + V_2 \tag{1.16}$$

1.10 Electric potential energy of 3 charges

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1Q_2}{r_{12}} + \frac{Q_2Q_3}{r_{23}} + \frac{Q_1Q_3}{r_{13}}\right) \tag{1.17}$$

1.11 Potential and field on the axis of a dipole

$$V = \frac{p}{4\pi\varepsilon_0 x^2}; \tag{1.18}$$

$$E_x \equiv -\frac{\partial V}{\partial x} = \frac{p}{2\pi\varepsilon_0 x^3} \tag{1.19}$$

where dipole moment $p = Qa$ with a being the charge separation. When defined as a vector, **p** points from – to +.

1.12 Off axis dipole field

The x-component is

$$E_x = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1) \tag{1.20}$$

See Question 6 on Problem Sheet 1 for the y- and z-components.

1.13 Potential energy U of a dipole in a uniform electric field

$$U = -\mathbf{p}\cdot\mathbf{E} \tag{1.21}$$

1.14 Torque $\boldsymbol{\Gamma}$ on a dipole in a uniform electric field

$$\boldsymbol{\Gamma} = \mathbf{p}\times\mathbf{E} \tag{1.22}$$

1.15 Dipole moment of a general charge distribution

$$p_n = \sum_i Q_i n_i, \quad n = x, y, z \tag{1.23}$$

$$\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} = \sum_i Q_i\mathbf{r}_i \tag{1.24}$$

If $\sum_i Q_i = 0$ (i.e. zero net charge), **p** is independent of the choice of origin.

2 Lines, flux and Gauss's flux law

2.1 Element of electric flux

$$d\Phi_E = \mathbf{E}\cdot d\mathbf{S} \tag{2.1}$$

2.2 Total flux through a surface

$$\Phi_E = \iint_S \mathbf{E}\cdot d\mathbf{S} = \iint_S E_\perp dS \tag{2.2}$$

These objects are known as *surface* integrals. Note that a single (rather than a double) integral sign is sometimes used.

2.3 Gauss's flux law

$$\oiint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\sum_i Q_i \tag{2.3}$$

The flux through a closed surface = the total charge enclosed $\times\varepsilon_0^{-1}$.

3 Electric field calculations

3.1 Foundation principle

The following formulae for the field and potential always work, although in simple cases one can use Gauss's Flux Theorem to avoid using them.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i\hat{\mathbf{r}}_i}{r_i^2} \tag{3.1}$$

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{r_i} \tag{3.2}$$

3.2 Spherical shell

radius a, total charge Q

$$E = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r^2} & (r > a) \\ 0 & (r < a) \end{cases} \tag{3.3}$$

3.3 Spherical distribution

radius a, uniform charge density ρ

$$E = \begin{cases} \frac{\rho a^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2} & (r > a) \\ \frac{\rho r}{3\varepsilon_0} & (r < a) \end{cases} \tag{3.4}$$

3.4 Infinite line

linear charge density λ, at distance r

$$E = \lambda/2\pi\varepsilon_0 r \tag{3.5}$$

3.5 Infinite plane sheet

areal charge density σ

$$E = \frac{\sigma}{2\varepsilon_0} \tag{3.6}$$

3.6 Circular ring

linear charge density λ, radius a, at distance z on axis

$$E = \frac{a\lambda}{2\varepsilon_0}\frac{z}{(a^2+z^2)^{3/2}} \to \frac{a\lambda}{2\varepsilon_0 z^2} \equiv \frac{Q}{4\pi\varepsilon_0 z^2} \text{ as } z \to \infty \text{ (as it should)!} \tag{3.7}$$

3.7 Circular disk

areal charge density σ, radius a, at distance z on axis

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right) \to \frac{\sigma}{2\varepsilon_0} \text{ as } z \to 0 \text{ (as it should)!} \tag{3.8}$$

3.8 Outside an infinite plane conductor

areal charge density σ

$$E = \frac{\sigma}{\varepsilon_0} \tag{3.9}$$

3.9 Inside a conductor under static conditions

$$E = 0; \quad V = \text{constant} \tag{3.10}$$

4 Capacitance

$$C = Q/V \tag{4.1}$$

4.1 Isolated sphere

radius a

$$C = 4\pi\varepsilon_0 a \tag{4.2}$$

4.2 Two concentric spheres

radii a and ($b > a$)

$$C = 4\pi\varepsilon_0 a \left(\frac{b}{b-a}\right) \tag{4.3}$$

4.3 Parallel plate capacitor

plate area A, plate separation s

$$C = \frac{\varepsilon_0 A}{s} \tag{4.4}$$

4.4 Stored energy in capacitor

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \tag{4.5}$$

4.5 Stored energy per unit volume

$$u = \frac{U}{volume} = \frac{1}{2}\varepsilon_0 E^2 \tag{4.6}$$

Adapted from Prof. G.H.C. New's original

5 Drude model for conduction

5.1 Drift velocity

$$\bar{v} = \frac{eE\lambda}{2m\bar{c}} \tag{5.1}$$

where λ is the mean path between collisions, e and m are the electronic charge and mass, and $\bar{c}$ is the mean thermal velocity.

5.2 Current and drift velocity

$$I = Ne\bar{v}A \tag{5.2}$$

where N is the density of free electrons (m^{-3}), and A is the cross-sectional area of the conductor. Note that the direction of I is opposite to that of $\bar{v}$ although (5.2) deals with magnitudes only.
One can equally write the equation in terms of the current density j (A/m^2) namely

$$j \equiv I/A = Ne\bar{v}$$

5.3 Conductivity

The conductivity σ of a material is defined from the equation

$$\mathbf{j} = \sigma\mathbf{E} \tag{5.3}$$

The resistivity ρ is the reciprocal of σ. According to this simple model of conduction

$$\sigma = \frac{Ne^2\lambda}{2m\bar{c}}$$

Since $E = V/L$ where V is the potential difference across a conductor and L is its length, it follows from (5.3) that

$$V = I\frac{L}{\sigma A} = IR \tag{5.4}$$

which is Ohm's Law. Clearly σ is in $(\Omega\text{m})^{-1}$ and ρ is in Ωm.

6 Introduction to Magnetism

6.1 Lorentz Force Law

The force on a particle of charge q moving at velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.1}$$

6.2 Force on a current element

$$d\mathbf{F} = Id\mathbf{s} \times \mathbf{B} \tag{6.2}$$

where $d\mathbf{s}$ is the length in the direction of positive current flow.

6.3 Torque on a rectangular current loop in a magnetic field

$$\mathbf{\Gamma} = I\mathbf{A} \times \mathbf{B} \tag{6.3}$$

where A is the area of the loop. The equation embodies the sign convention that, looking in the direction of the vector $\mathbf{A}$, a clockwise current counts as positive.

6.4 Hall Effect

For a conductor carrying current I in a transverse magnetic field B, the Hall voltage is

$$V_{Hall} = \frac{wIB}{NeA} \tag{6.4}$$

where w is the width of the conductor (the dimension across which the voltage is registered), A is its cross-sectional area, and N is the free electron density.

7 Magnetic Field Calculations

7.1 Magnetic flux

$$\Phi_B = \iint \mathbf{B} \cdot d\mathbf{S}$$

7.2 Gauss's Flux Law for magnetism

$$\oiint \mathbf{B} \cdot d\mathbf{S} = 0 \tag{7.1}$$

i.e. the flux of $\mathbf{B}$ through any closed surface is zero. This amounts to a statement that free magnetic poles don't exist.

7.3 Biot-Savart Law (the magnetic field generated by a current element)

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \tag{7.2}$$

where r is the distance from the current element $Id\mathbf{s}$ to the point of observation, and $\hat{\mathbf{r}}$ is the unit vector in this direction.

7.4 Mutual interaction of two current elements

By combining (6.2) and (7.2), the force on element 2 is found to be

$$d^2\mathbf{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi} \frac{d\mathbf{s}_2 \times (d\mathbf{s}_1 \times \hat{\mathbf{r}})}{r^2} \tag{7.3}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from 1 to 2.

7.5 Circular loop

At the centre of a loop of radius a

$$B = \frac{\mu_0 I}{2a} \tag{7.4}$$

where B points in the direction in which the current is seen to flow in a clockwise direction.
On the axis of the loop at a distance z from the centre

$$B = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \tag{7.5}$$

7.6 Magnetic field from an infinite straight wire

The radial component of the field is zero. At a distance r from the wire, the azimuthal component is

$$B = \frac{\mu_0 I}{2\pi r} \tag{7.6}$$

Looking along the wire in the direction of the current, the field lines circle the wire in a clockwise direction.

7.7 Force per unit length between two parallel wires carrying currents

$$f = \frac{\mu_0 I_1 I_2}{2\pi r} \tag{7.7}$$

where r is the separation of the wires. If the currents are flowing in the same direction the force is attractive. The equation is obtained by combining (7.6) and (6.2).

7.8 Ampère's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{7.8}$$

where I is the current flowing though any surface bounded by the circuit defined by the loop integral. Current flow is positive in the direction in which the loop integral is seen as clockwise.

7.9 Toroidal coil (on axis)

$$B = \frac{\mu_0 N I}{2\pi r} = \mu_0 n I \tag{7.9}$$

where N is the number of turns and $2\pi r$ is the circumference of the toroid; n is the number of turns per unit length.

7.10 Long solenoid

$$B = \mu_0 n I \tag{7.10}$$

where n is the number of turns per unit length—basically the same result as for a toroid.

Adapted from Prof. G.H.C. New's original

8 Electric and Magnetic Fields in Material Media

8.1 Magnetic dipole moment

The magnetic dipole moment of a current loop of vector area **A** is

$$\mathbf{m} = I\mathbf{A} \tag{8.1}$$

Torque on a magnetic dipole in a field (see Eq.(6.3)) is

$$\mathbf{\Gamma} = \mathbf{m} \times \mathbf{B}; \tag{8.2}$$

The potential energy of a magnetic dipole in a **B** field is

$$U = -\mathbf{m} \cdot \mathbf{B} \tag{8.3}$$

For the electrostatic analogues to Eqs. (8.2) and (8.3), see Eqs. (1.22) and (1.21), respectively.

8.2 Properties of Dielectrics and the Dielectric Constant

In dielectric materials (insulators), electrons are bound to their parent atoms. However, under the action of an electric field, the charge distribution within an atom is distorted, and an *atomic dipole moment* **p** is induced. The overall effect is to create a large-scale "polarisation" of the medium defined as $\mathbf{P} = N\mathbf{p}$ where N is the atomic number density and the *polarisation vector* **P** is the dipole moment per unit volume; note that the dimensions of **P** are charge per unit area.

If an external electric field E_0 is applied to an insulator, the field E within the medium is given by

$$E = E_0 - E_{induced}$$

where $E_{induced}$is the reverse field arising from the polarisation. Written in vector form, this equation reads

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P} =}{\varepsilon_0} = \frac{\mathbf{E}_0}{\varepsilon_r} \tag{8.4}$$

where ε_r is the *dielectric constant* of the medium.

Note that $\mathbf{E}_0$ is traditionally replaced by $\mathbf{D}/\varepsilon_0$, where **D** is known as the *electric displacement vector*. The final step in Eq. (8.4) indicates that the dielectric constant is the factor by which the external field is reduced within the dielectric.

Rearranging Eq. (8.4) yields

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_r\varepsilon_0\mathbf{E} \tag{8.5}$$

which is the form normally found in textbooks.

8.3 The Magnetic Analogue

The (albeit imperfect) magnetic analogue to Eq. (8.4) is

$$\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} = \mu_r\mathbf{B}_0 \tag{8.6}$$

where $\mathbf{B}_0$ is the field from external currents, **M** is the magnetisation vector, and μ_r is the relative magnetic permeability of the medium. The vector $\mathbf{B}_0$ is traditionally written $\mu_0\mathbf{H}$, where **H** (in convenient units of A/m) is known as the *magnetic intensity* or the *magnetic field*, or the *H*-field (to distinguish it from **B** , which some books call the *magnetic induction field* to distinguish it from **H**)!!

Rearranging Eq. (8.6) now yields

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_r\mu_0\mathbf{H} \tag{8.7}$$

In free space, $\mu_r = 1$ and $\mathbf{B} = \mu_0\mathbf{H}$.

Note that **M** is small and negative in *diamagnetic* media, and small and positive in *paramagnetic* media; $\mu_r \sim 1$ in these cases. In *ferromagnetic* media, **B** is large and positive.

Adapted from Prof. G.H.C. New's original

9 Time varying currents and electromagnetic induction

9.1 Electromotive Force and EM Induction

9.1.1 Faraday's Law of Electromagnetic Induction

$$V_{EMF} = (-)\frac{d\Phi_B}{dt} \tag{9.1}$$

where V_{EMF} is the EMF induced in a circuit, and Φ_B is the magnetic flux through the circuit (see Section 7). The minus sign embodies a sign convention.

9.1.2 Self inductance

The self-inductance of a component (or circuit) carrying current I is defined as

$$L = \frac{\Phi_B}{I} \tag{9.2}$$

where Φ_B is the flux linking the circuit

The EMF induced in an inductor by a changing current is

$$V_i = \pm L\frac{dI}{dt} \tag{9.3}$$

where the sign is sometimes written + and sometimes - depending on the convention. Using the + sign brings the formula for the voltage across an inductor into line with the standard formula $V = IR$ for a resistor.

The self-inductance of a long solenoid is

$$L = \frac{\mu_0 A N^2}{\ell} \tag{9.4}$$

where A is the cross-sectional area, N the number of turns, and ℓ the length of the solenoid.

9.1.3 Mutual Inductance

The mutual inductance of two coils wound on a single solenoid is

$$M = \frac{\mu_0 A N_1 N_2}{\ell} = \sqrt{L_1 L_2} \tag{9.5}$$

Adapted from Prof. G.H.C. New's original