Information needed for this Problem sheet:

- $G = 6.673 \times 10^{-11}$; N m^2 kg^{-2};
- $\varepsilon_0 = 8.854 \times 10^{-12}$F m^{-1};
- $e = 1.602 \times 10^{-19}$C;
- $m_e = 9.109 \times 10^{-31}$kg;
- $g = 9.807$ m s^{-1};
- $a_0 = 5.29 \times 10^{-11}$m
- Atomic number density of Cu = 8.5×10^{28} m^{-3}.

Key point

Sometimes it's easier to find the electrostatic potential V first, and to differentiate to get the electric field $\boldsymbol{E}$; sometimes it's easier to start with $\boldsymbol{E}$, and integrate for V. It's often not easy to see in advance which will be the easier strategy!

Some simple exercises to get you going

(Answers are given at the end)

1. Calculate the ratio of the electrostatic to the gravitational force between two electrons. Why is the ratio independent of their separation?
2. Consider two 1 cm^3 blocks of copper 10 m apart. If, in each block, one electron is removed from one atom in a million, find the magnitude of the repulsive force between the blocks. Roughly how many people could the force support?
3. A 0.8nC charge lies at (1,1,0) and a -1.2nC charge at (1,2,0). Find (a) the potential energy, (b) the (electrostatic) potential at (3,3,0), (c) the potential at $(0,0,\sqrt{2})$. Distances are in millimetres.
4. Find the electric field if the electrostatic potential is (a) $V = \alpha(x+y)$ and (b) $V = \beta xyz$, where α and β are constants.
5. A charge +Q and two charges $-\alpha Q$ (where $\alpha > 0$) are situated at the corners of an equilateral triangle. For what values of α is the potential energy negative?
6. An electric dipole consists of charges $\pm 2.0\times 10^{-8} C$ separated by 1 cm. What is the magnitude of the electric field at the centre of the dipole?
7. Sketch the electric field lines from charges -Q and +2Q a distance a apart. Consider the following points:
   (a) Will the field pattern be symmetric?
   (b) What does the field pattern look like very close to each charge?
   (c) At what distance from the $-Q$ charge is the electric field zero?
   (d) What does the field pattern look like a long distance ($\gg a$) from the charges?

   You might want to look at the field line configuration using one of the JAVA applets provided on the course website.

Problems

1. Find the magnitude of the electric field at a distance of one Bohr radius (a_0) from a proton; this is the field experienced by an electron in the first Bohr orbit (the ground state) of a hydrogen atom.

   Calculate the potential energy of the electron and proton taking the zero of potential energy to be at infinite separation. Express your answer (a) in Joules and (b) in electron volts (eV).

   Now assume the electron orbits the proton in a circular path of radius a_0. Find (a) the orbital velocity of the electron, (b) the ratio of potential to kinetic energy, and (c) the total energy of the electron (= the ground state energy of hydrogen).

   [Ignore any radiation of energy associated with the electron's acceleration. One of the Bohr postulates is precisely that there isn't any!]

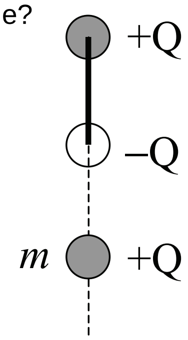
2. Charges are located at each of the four corners of a square of side a. Find the potential energy for each of the following configurations:

   (a) Four identical charges $+Q$;
   (b) Two charges $+Q$ and two charges $-Q$, with charges of like sign at opposite corners;
   (c) Two charges $+Q$ and two charges $-Q$, with charges of like sign at adjacent corners.

   Could the charges be described as bound by the electrostatic forces in any of the three cases and, if so, in which?

3. Consider a particle of mass m and charge $+Q$ situated directly beneath a dipole as shown in the diagram. If the three charges have equal magnitudes of 3 nC, and are equally spaced 1 cm apart, find the value of m for which the particle is held in equilibrium under gravity.

   Is the equilibrium stable or unstable?



4. In connection with the energy stored in a capacitor, we showed in lectures that the mutual potential energy of three charges Q_1, Q_2 and Q_3 is

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1 Q_2}{r_{12}} + \frac{Q_1 Q_3}{r_{13}} + \frac{Q_2 Q_3}{r_{23}}\right)$$

   where r_{12} is the separation of Q_1 and Q_2 etc. Show that the result may be written in the form

$$U = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$$

   where V_i is the potential at Q_i due to the other two charges.

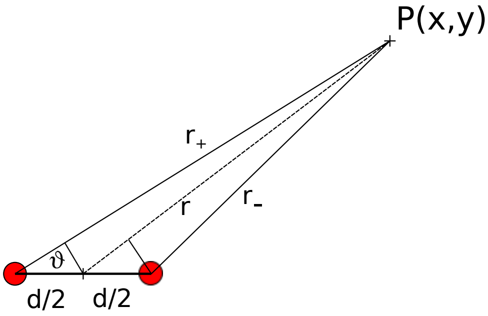
5. Show that, in the far field approximation ($x \gg d$), the electric potential of a dipole formed by two charges $\pm Q$ situated at $(\pm d/2,\ 0,\ 0)$ on this x axis is given by

$$V = \frac{p}{4\pi\varepsilon_0 x^2}$$

   where the dipole moment $p = Qd$. Obtain the associated electric field by differentiation.

   Compare this way of solving the problem with that obtained in Question 2 of Classwork 1.

6. Obtain expressions in the far field approximation ($r \gg d$) for the electric potential and electric field at a point P, distance r (measured from the mid-point of the dipole) away from a dipole formed by two charges $\pm Q$ situated at $(\pm d/2,\ 0,\ 0)$ (see figure below).



You may like to follow steps (a) to (c) initially:

(a) Obtain an approximate expression for r_+ and r_- as a function of r.

(b) Use an exact formula to obtain the electric potential due to both charges at P with distances r_+ and r_-.

(c) Use a series expansion on the distances assuming that $r \gg d$ to obtain

$$V \cong \frac{px}{4\pi\varepsilon_0 r^3}$$

(d) Obtain E_x by differentiating V and not forgetting that $r = f(x)$:

$$E_x = -\frac{\partial V}{\partial x} = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$

where θ is the angle between the mid-point and the x-axis (see figure).

(e) Obtain the corresponding expressions for E_y and E_z.

(f) Show that $E_y = E_z = 0$ on the y-axis.

(g) Show that the locus of points for which $E_x = 0$ in the far field is a cone whose vertex angle is $2\cos^{-1}(1/\sqrt{3})$.

Answers to Exercises

1. 4.16×10^{42}.

2. 26 people (assuming an average weight of 65 kg = 143 lb = just over 10 stone).

3. (i) -8.63 μJ, (ii) -2.28 kV, (iii) -482 V.

4. (i) $\mathbf{E} = -\alpha(\mathbf{i} + \mathbf{j})$; (ii) $\mathbf{E} = -\beta(yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})$

5. $0 < \alpha < 2$.

6. 1.44×10^7 V/m

7. The pattern is symmetric to rotation about the axis defined by the two charges, but not symmetric with respect to direction along the axis. Close to each charge, the lines are similar to those of isolated charges. Far away, the lines are similar to those for a single $+Q$ charge. $E = 0$ is at $x = a(1 + \sqrt{2})$

Information needed for this Problem sheet:

- $G = 6.673 \times 10^{-11}$ N m^{2}kg^{-2};
- $\varepsilon_0 = 8.854 \times 10^{-12}$F m^{-1};
- $e = 1.602 \times 10^{-19}$C;
- $m_e = 9.109 \times 10^{-31}$kg;
- $g = 9.807$ m s^{-1};
- $a_0 = 5.29 \times 10^{-11}$m

Some simple exercises to get you going

(Answers are given at the end)

1. Find the electric flux through a closed surface surrounding the charges -2.5μC, -1.3μC, +0.8μC and +3.2μC.

2. The charge density (Coulombs per cubic metre) within a sphere of radius a varies as $\rho(r) = \beta(r/a)^n$ where β is a constant, n is an integer, and r is distance from the centre. Show that the *total* charge in the sphere is $Q = (4\pi\beta a^3)/(n+3)$

3. A copper plate 1 mm thick is placed in a strong external electric field $E_{ext} = 10^6$V/m normal to the surface. If copper has 8.5×10^{28} free electrons per m^3, what proportion of these migrate to the surface to maintain zero internal field? Might there be a shortage of electrons as the plate is made thinner?

4. Using your lecture notes as little as possible (preferably not at all!) and Gauss's Flux Law in every case, reproduce the following electric field calculations done in the lectures:

   (a) Show that the electric field of a uniformly charged spherical shell of radius a carrying total charge Q is

   $$E = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r^2} & (r > a) \\ 0 & (r < a) \end{cases}$$

   (b) Show that the electric field at a distance x from a infinite line charge of linear charge density λC/m is

   $$E = \frac{\lambda}{2\pi\varepsilon_0 x}$$

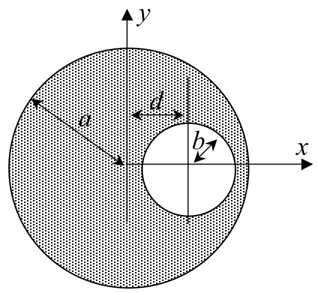
   (c) Show that the electric field adjacent to a infinite charge sheet of areal charge density σ C/m^2 is

   $$E = \frac{\sigma}{2\varepsilon_0}$$

Problems

1. This is a well-known electrostatics problem, which looks horrendous at first sight, but is in fact easily solved using the Principle of Superposition. Refer to the hint at the end of the sheet if you want a clue! The result is remarkable.

   Consider a uniformly charged sphere of radius a carrying charge density ρ from which a spherical section of radius b ($< a$) has been removed to leave a hollow cavity. The centre of the cavity is at d ($< a - b$) from the centre of the larger sphere (see diagram).



Show that the electric field is $\mathbf{E} = (\rho d/3\varepsilon_0)\hat{\mathbf{i}}$ everywhere within the cavity.

2. Consider a thin circular disk of radius a lying in the $x-y$ plane with its centre at the origin and carrying a uniform charge density σ. Find the z-component of the field from a thin annular ring directly, and then integrate over the disk. Check for the special case of $\alpha \to \pi/2$ (α being the angle at which the rim of the disk is seen from the observation point on axis) the result yields the electric field of an infinite sheet. Now check what happens when $\alpha \to 0$. Why does this limit not yield the electric field due to a point charge. How can you rectify the problem?

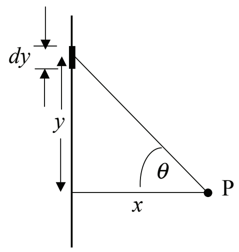
3. In the lectures, Gauss's Flux Law was used to show that the electric field a distance x from an infinite line charge lying along the y-axis is

$$E(x) = \frac{\lambda}{2\pi\varepsilon_0 x}$$

where λ (C/m) is the charge per unit length.

Now obtain this answer by direct integration. This is not only a good exercise; if you want to find the field from a line charge of *finite* length, you have to do it this way!

The question leads you through the calculation in easy stages. Start by finding the contribution from an element dy and go on to integrate over all such elements.



   (a) Show that the magnitude of the contribution to the electric field at P (see diagram) from the element dy at y is
$$dE = \frac{\lambda \cos^2\theta \, dy}{4\pi\varepsilon_0 x^2}$$

   (b) Show that the x-component of the field from this element, *taken together with the corresponding element at* $-y$ is
$$dE_x = \frac{\lambda \cos^3\theta \, dy}{2\pi\varepsilon_0 x^2}$$

   (c) The problem can now be solved by integrating either over θ from 0 to $\pi/2$, or over y from 0 to ∞. To take the former route, first show that $dy = (xd\theta)/\cos^2\theta$

   (d) Perform the integration to obtain the result.

   (e) If you're on a roll, try the other integration option suggested in (3c).

(f) If you try to solve this problem by working out the potential first, you will get into a mess. You might want to think why this is!

This standard integral will come in handy:

$$\int \frac{dy}{(a^2+y^2)^{3/2}} = \frac{y}{a^2(a^2+y^2)^{1/2}}$$

4. A capacitor is constructed by sandwiching a thin sheet of plastic (dielectric constant 3.5) between two sheets of metal foil. All three sheets are 0.05 mm thick and measure 1 m by 3 cm. Find the capacitance.

   A second plastic sheet identical to the first is now placed on top of the upper sheet of foil and the strip rolled up tightly like a rolled carpet. How is the capacitance changed? Some commercial capacitors are fabricated in this way.

   [Hint: The dielectric constant ε_r will be covered later in the course. For the moment, simply make the replacement $\varepsilon_0 \rightarrow \varepsilon_r\varepsilon_0$ in the equations to get the result.]

5. (a) A small sphere of radius a carries charge q distributed uniformly over the surface. From the formula for the capacitance and the expression for the energy stored in a capacitor, show that the stored energy is

$$U = \frac{q^2}{8\pi\varepsilon_0 a}$$

   (b) Obtain the same result by working out the total energy stored in the electrostatic field. Use the fact that the energy density (J/m^2) in an electrostatic field in vacuo is $(\varepsilon_0 E^2)/2$.

   (c) The electron appears to behave in most respects as a point particle. However, if one assumes that it is actually a small sphere, one can estimate its radius by equating the stored electrical energy to its rest energy mc^2. Show that this predicts an electron radius of

$$R = \frac{e^2}{8\pi\varepsilon_0 mc^2}.$$

   Put in the numbers to calculate the value of R.

   (The expression for R is half the “classical electron radius” as usually defined. However, the problem of the electron’s size is complicated, and the interpretation of R should not be taken too literally.)

Answers to Exercises

1. 2.26×10^4Vm;
3. 6.5×10^{-13}

[HINT FOR QUESTION 1: The field you want to find is the field from the complete larger sphere MINUS the field from the piece that has been removed.]

Peter Török

Information needed for this Problem sheet:

- $\varepsilon_0 = 8.854 \times 10^{-12}$F m^{-1};
- $\mu_0 = 4\pi \times 10^{-7}$H m^{-1};
- $e = 1.602 \times 10^{-19}$C;
- $m_e = 9.109 \times 10^{-31}$kg;
- Atomic number density of Cu = 8.5×10^{28}m^{-3}.
- The resistivity of copper is $1.7 \times 10^{-8}\Omega$m.

Some simple exercises to get you going (Answers are given at the end)

1. A piece of copper wire radius 0.5 mm carries a uniformly distributed current of 1 A. Find the electron drift speed in the wire.

   [For comparison, the thermal speed of the electrons at 20°C is of the order of 10^5 m/s.]

2. If the wire in Part 1 is 100 m long, calculate the energy dissipated in the wire in one minute.

3. A straight copper bar with *rectangular* cross section is aligned along the z-axes and carries a current of 3A in this direction. A magnetic field B of 1 T is applied in the x-direction, perpendicular to two sides of the bar that are separated by 1.5 mm. In what direction does the Hall effect voltage appear, and what is its magnitude?

4. Find the force per unit length between two long straight parallel wires, 5 cm apart, carrying currents of 1 kA in opposite directions. Is the force attractive or repulsive?

5. What is the magnitude of the magnetic field experienced by each wire due to the other in Part 4?

6. Find the current in a 20 turn coil of radius 8 cm needed to produces a magnetic field of $B = 3.0 \times 10^{-3}$ T at its centre.

7. A power line running east-west 20 m above the ground carries a current in an easterly direction. What is the direction of the associated magnetic field? Find the current in the wire for which the magnitude of the magnetic field at ground level directly below the line is roughly the same as that of the earth's field ($\sim 10^{-4}$T).

Problems

1. A straight piece of wire running along the y-axis between $y = \pm L/2$ forms part of an electrical circuit carrying current I (see Fig. 1 below). Show that its contribution to the magnetic field at P, a distance a from the origin in the $x - z$ plane, is

$$B = \frac{\mu_0 I \cos\theta}{2\pi a}$$

   HINT: Use a simple extension to the proof for the *infinite* straight wire given in the lectures. There, we integrated over an angle θ, but you can equally integrate over y using the standard integral:

$$\int \frac{dy}{(a^2 + y^2)^{3/2}} = \frac{1}{a^2} \frac{y}{\sqrt{(a^2 + y^2)}}.$$

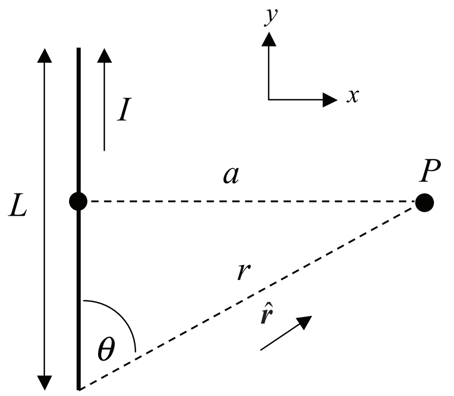


Fig. 1

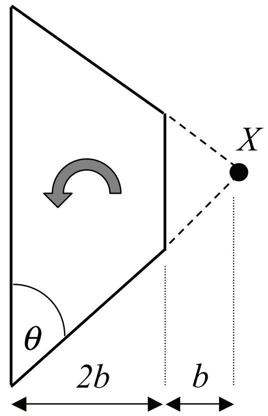


Fig. 2

2. A current of 3 A flows anticlockwise round the trapezoidal circuit shown in fig. 2. If b = 4 mm and $\theta = 45°$, find the magnitude and direction of the magnetic field at X.

3. A thin circular disk of insulating material of thickness t and the radius $a(\gg t)$ carries uniform charge density ρ. If the disk rotates at angular velocity ω about its axis (see Fig. 3), show that the magnetic field close to the centre of the disk (arrowed) is

$$B = \frac{\mu_0 \rho t \omega a \mathbf{k}}{2}$$

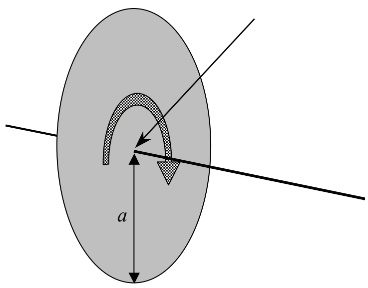


Fig. 3

4. A square current loop of side a centred at the origin has its four corners at $x, y = \pm a/2$. The loop carries a current I in a clockwise sense for an observer looking in the $+z$ direction. Show that the magnetic field on axis at $(0, 0, z)$ where $z \gg a$ is

$$\mathbf{B} = \left(0,\ 0,\ \frac{\mu_0 m}{2\pi z^3}\right)$$

where m $(= Ia^2)$ is the magnetic dipole moment of the loop. Compare the answer with the analogous result for an *electric* dipole.

[HINT: Use the Biot-Savart Law to find the field from a single side. Don't forget to resolve along the z-axis, and to use the fact that $z \gg a$. Since a is small, you don't need to integrate. Multiply by 4 to get the result from all four sides.]

4. (a) Show that a particle of mass M and charge q moving in a circular orbit with (vector) angular momentum $\mathbf{L}$ is equivalent to a magnetic dipole with a dipole moment $\mathbf{m} = q\mathbf{L}/2M$.

(b) In the Bohr model of the hydrogen atom, the orbital angular momentum of the electron is restricted to integer multiples of $\hbar$ (= $h/2\pi$). Calculate the magnitude of the magnetic dipole moment of the electron in the ground state of the hydrogen atom. [This value is termed the Bohr magneton, and is usually denoted μ_B.]

(c) The potential energy of a magnetic dipole **m** in a magnetic field is $U = -\mathbf{m} \cdot \mathbf{B}$. If a magnetic field is applied to a collection of hydrogen atoms, one might expect the atoms to line up in the direction of lowest potential energy (i.e. with their dipole moments aligned in the direction of B). Calculate the potential energy difference between orientations parallel and antiparallel to the magnetic field for a single hydrogen atom in a magnetic field of 10 T (which is about the maximum steady magnetic field that can be generated in the laboratory). Compare this with the average thermal kinetic energy of hydrogen atoms ($\sim \frac{3}{2}kT$at room temperature). What conclusion can be drawn?

5. Two identical circular coils (radius a, each with N turns) are arranged coaxially (on the x-axis) with their centres at $x = \pm L$ (see diagram). Both carry a current I in the same sense. Show that the magnetic field at an arbitrary point on the x-axis is

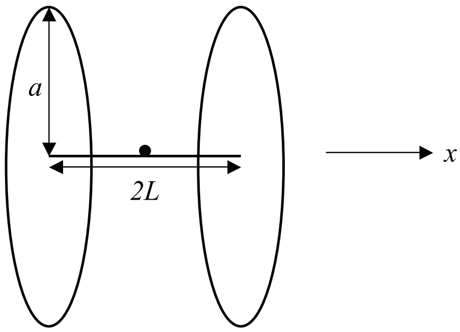


Fig. 4

$$B = \frac{\mu_0 N I a^2}{2} \left\{ \frac{1}{[(L+x)^2 + a^2]^{3/2}} + \frac{1}{[(L-x)^2 + a^2]^{3/2}} \right\}$$

Differentiate to show that

$$\frac{dB}{dx} = \frac{-3\mu_0 N I a^2}{2} \left\{ \frac{L+x}{[(L+x)^2 + a^2]^{5/2}} - \frac{L-x}{[(L-x)^2 + a^2]^{5/2}} \right\}$$

$$\frac{d^2B}{dx^2} = \frac{3\mu_0 N I a^2}{2} \left\{ \frac{4(L+x)^2 - a^2}{[(L+x)^2 + a^2]^{7/2}} - \frac{4(L-x)^2 - a^2}{[(L-x)^2 + a^2]^{7/2}} \right\}$$

Check that $dB/dx = 0$ at $x = 0$, which should be obvious by symmetry. For what value of L is d^2B/dx^2 zero?

[COMMENT: This question is very simple in principle, but it's algebraically laborious.]

Answers to Exercises

1. $9.4 \times 10^{-}5$m/s
2. 130 J
3. 0.147 μV
4. 4.0 N/m repulsive
5. 4×10^{-3} T
6. 19.1 A
7. North. 10 kA

Peter Török

1st year E & M **Problem sheet X**

Harder problems for a greater challenge

Note: I might be adding problems to this file as we progress through the course so please come back and check again.

1. The field on the axis of a dipole in the far-field approximation was shown in Classwork 1 to be

$$E_x = \frac{p}{2\pi\varepsilon_0 x^3}$$

By including higher-order terms in the binomial expansion, show that a more accurate expression for the field is

$$E_x = \frac{p}{2\pi\varepsilon_0 x^3}\left(1 + \frac{a^2}{2x^2} + \frac{3a^4}{16x^4}\right)$$

where a is the charge separation. Hence make a rough estimate of the smallest value of x for which the simpler formula is good to 10 percent. (Use only the a^2 term for this last part.)

[HINT: You will need one or other of the following binomial expansions, depending on whether you choose to work out the potential first and differentiate, or go direct to the field (see also Problem 5 on Problems Sheet 1):

$$(1 \pm \delta)^{-1} = 1 \mp \delta + \delta^2 \mp \delta^3 + \delta^4 \mp \delta^5 + \ ...$$

$$(1 \pm \delta)^{-1} = 1 \mp 2\delta + 3\delta^2 \mp 4\delta^3 + 5\delta^4 \mp 6\delta^5 + \ ... \ ]$$

2. In Problem 3 of Problem Sheet 2, you found the electric field of an *infinite* line charge. Consider now the case where the line charge is of *finite* length in the interval $-L < y < +L$. Prove that the potential at a distance x from the line at $y = 0$ is

$$V(x) = \frac{\lambda}{2\pi\varepsilon_0} \log\left\{\frac{L + \sqrt{L^2 + x^2}}{x}\right\}$$

[HINT: Use the standard integral

$$\int \frac{dy}{\sqrt{y^2 + a^2}} = \log\{y + \sqrt{y^2 + a^2}\}.$$

If you integrate from 0 to L and multiply by 2, the quoted result is easily retrieved. If you integrate from -L to +L, you appear to get a *different* answer; fortunately, it's the *same* answer in a different algebraic form.]

3. By differentiating $V(x)$ in the previous question, prove that

$$E(x) = \frac{\lambda}{2\pi\varepsilon_0 x}\left(\frac{L}{\sqrt{L^2 + x^2}}\right)$$

Note that $E(x) \to \lambda/2\pi\varepsilon_0 x$ as $L/x \to \infty$ as of course it should!

[NOTE: This is straightforward in principle, but algebraically awkward in practice.]

4. The electrostatic potential and field outside a uniformly charged spherical shell is easily shown by Gauss's Flux Law to be the same as if all the charge were concentrated at the centre. Prove this result <u>*without*</u> using Gauss's Flux Law.

5. As a preliminary to the next question, try the analogous question in electrostatics.

   An electric dipole moment located at the origin is directed along the z-axis (i.e. $\mathbf{p} = p\mathbf{k}$). Show that at large distances from the origin in the $z = 0$ plane, the electric field is

$$\mathbf{E} = \left(0,\ 0,\ -\frac{p}{4\pi\varepsilon_0 r^3}\right) \quad \text{where} \quad r^2 = x^2 + y^2$$

6. A square current loop of side a centred at the origin has its four corners at $x, y = \pm a/2$.

   The loop carries a current I in a clockwise sense for an observed looking in the $+z$ direction. Show that the magnetic field at $(x, 0, 0)$ where $x \gg a$ is

$$\mathbf{B} = \left(0,\ 0,\ -\frac{\mu_0 m}{4\pi x^3}\right)$$

   where m (= Ia^2)is the magnetic moment of the loop.

   [HINT: Take the four sides in pairs, and don't assume that only two contribute!]

1st Year Electricity & Magnetism
ANSWERS to Problem Sheet 1

Exercises

A. $\left(\dfrac{e^2}{4\pi\varepsilon_0 r^2}\right)\Big/\left(\dfrac{Gm^2}{r^2}\right)=\dfrac{e^2}{4\pi\varepsilon_0 Gm^2}=4.16\times10^{42}$

The result is independent of *r* because both the electrostatic and the gravitational forces obey an inverse square law.

B. Charge in each block $Q=1.6\times10^{-19}\times8.5\times10^{28}\times10^{-6}\times10^{-6}=1.36\times10^{-2}\,\text{C}$;

$\dfrac{Q^2}{4\pi\varepsilon_0 10^2}=1.66\times10^4$ N

A 65 kg (=10 st 3 lb) person weights 637 N, so roughly 26 people of this weight could be supported.

C. (i) The charges are 1 mm apart so the potential energy is $-\dfrac{-1.2\times0.8\times10^{-18}}{4\pi\varepsilon_0\times10^{-3}}=8.63\,\mu\text{J}$.

(ii) Distances from (3,3,0) to (1,1,0) and (1,2,0) are $\sqrt{8}$ and $\sqrt{5}$ respectively so

$$V=\frac{10^{-6}}{4\pi\varepsilon_0}\left(\frac{0.8}{\sqrt{8}}-\frac{1.2}{\sqrt{5}}\right)=-2.28\,\text{kV}$$

(ii) Distances from (0,0,$\sqrt{2}$) to (1,1,0) and (1,2,0) are 2 and $\sqrt{7}$ respectively so

$$V=\frac{10^{-6}}{4\pi\varepsilon_0}\left(\frac{0.8}{2}-\frac{1.2}{\sqrt{7}}\right)=-482\,\text{V}.$$

D. (i) If $V=\alpha(x+y)$, $E_x=-\dfrac{\partial V}{\partial x}=-\alpha=-\dfrac{\partial V}{\partial y}=E_y$; $E_z=0$; $\boldsymbol{E}=-\alpha(\boldsymbol{i}+\boldsymbol{j})$.

(ii) If $V=\beta xyz$, $E_x=-\beta yz$; $E_y=-\beta xz$; $E_z=-\beta xy$;

$\boldsymbol{E}=-\beta(yz\boldsymbol{i}+zx\boldsymbol{j}+xy\boldsymbol{k})$.

E. Potential energy $=\dfrac{Q^2}{4\pi\varepsilon_0}(-\alpha-\alpha+\alpha^2)$; $(\alpha^2-2\alpha)$ is zero at $\alpha=0$ and $\alpha=2$, and negative in between these values.

F. $E=\dfrac{2Q}{4\pi\varepsilon_0(d/2)^2}=1.44\times10^7$ V/m.

G. The pattern is symmetric to rotation about the axis defined by the two charges, but not symmetric with respect to direction along the axis. Close to each charge, the lines are similar to those of isolated charges. Far away, the lines are similar to those for a single +*Q* charge. The electric field is zero at $x=a(1+\sqrt{2})$ distance from the –Q charge.

Problems

1. $\dfrac{e}{4\pi\varepsilon_0 a_0^2} = 5.14\times10^{11}$ V/m; $\dfrac{-e^2}{4\pi\varepsilon_0 a_0} = -4.35\times10^{-18}\,\text{J} \Rightarrow -27.2\,\text{eV}$

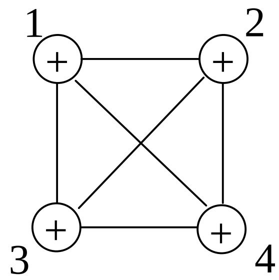
The value in “electron-volts” is found by dividing the energy in Joules by the elementary charge 1.602×10^{-19} C.

Orbital velocity $\text{v} = \sqrt{\dfrac{e^2}{4\pi\varepsilon_0 m a_0}} = 2.19\times10^6$ m s^{-1}

Kinetic energy $\dfrac{m\text{v}^2}{2} = \dfrac{e^2}{8\pi\varepsilon_0 a_0} = -\dfrac{1}{2}$potential energy.

So the total energy is $-2.17\times10^{-18}\,\text{J} = -13.6$ eV.

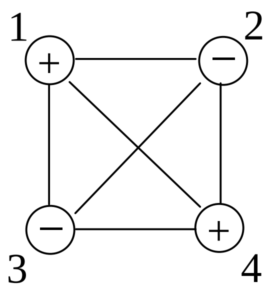
2(a)



$$V = \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a}+\frac{1}{a}+\frac{1}{a\sqrt{2}}+\frac{1}{a\sqrt{2}}+\frac{1}{a}+\frac{1}{a}\right) = \frac{Q^2}{4\pi\varepsilon_0 a}(\sqrt{2}+4)$$

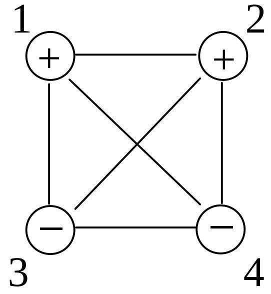
12 13 14 23 24 34

2(b)



$$V = \frac{Q^2}{4\pi\varepsilon_0}\left(-\frac{1}{a}-\frac{1}{a}+\frac{1}{a\sqrt{2}}+\frac{1}{a\sqrt{2}}-\frac{1}{a}-\frac{1}{a}\right) = \frac{Q^2}{4\pi\varepsilon_0 a}(\sqrt{2}-4)$$

2(c)



$$V = \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{a}-\frac{1}{a\sqrt{2}}-\frac{1}{a\sqrt{2}}-\frac{1}{a}+\frac{1}{a}\right) = \frac{Q^2}{4\pi\varepsilon_0 a}(-\sqrt{2})$$

Charges are bound in cases (b) and (c) since $V < 0$.

3. Upward force on mass $= \dfrac{Q^2}{4\pi\varepsilon_0 a^2} - \dfrac{Q^2}{4\pi\varepsilon_0 (2a)^2} = \dfrac{3Q^2}{16\pi\varepsilon_0 a^2} = mg$ to balance.

$$m = \frac{3Q^2}{16\pi\varepsilon_0 a^2 g} = \frac{3\times 9\times 10^{-18}}{16\pi\times 8.85\times 10^{-12}\times 10^{-4}\times 9.81} = 6.18\times 10^{-5}\,\text{kg}.$$ Unstable equilibrium.

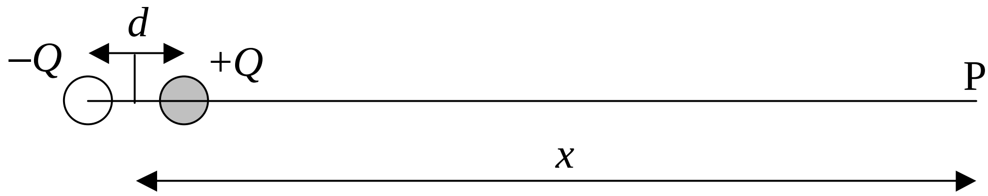
4.

$$V_1 = \frac{Q_2}{4\pi\varepsilon_0 r_{12}} + \frac{Q_3}{4\pi\varepsilon_0 r_{13}}$$

$$V_2 = \frac{Q_1}{4\pi\varepsilon_0 r_{12}} + \frac{Q_3}{4\pi\varepsilon_0 r_{23}} \qquad Q_1V_1 + Q_2V_2 + Q_3V_3 = 2U \text{ so } U = \frac{Q_1V_1 + Q_2V_2 + Q_3V_3}{2}$$

$$V_3 = \frac{Q_1}{4\pi\varepsilon_0 r_{13}} + \frac{Q_2}{4\pi\varepsilon_0 r_{23}}$$

5.



$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{x-d/2} + \frac{-Q}{x+d/2}\right) = \frac{Q}{4\pi\varepsilon_0 x}\left((1-d/2x)^{-1} - (1+d/2x)^{-1}\right)$$

$$\cong \frac{Q}{4\pi\varepsilon_0 x}\left(1+\frac{d}{2x}.................... - (1-\frac{d}{2x}.....................)\right) = \frac{Qd}{2\pi\varepsilon_0 x^2} = \frac{p}{2\pi\varepsilon_0 x^2}$$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{p}{4\pi\varepsilon_0}\left(\frac{-2}{x^3}\right) = \frac{p}{2\pi\varepsilon_0 x^3}$$

6. (a-c)

$$r_\pm \cong r\left(1 \mp \frac{d\cos\vartheta}{2r}\right), \quad V \cong \frac{Q}{4\pi\varepsilon_0 r}\left[\left(1-\frac{d\cos\vartheta}{2r}\right)^{-1} - \left(1+\frac{d\cos\vartheta}{2r}\right)^{-1}\right] \cong \frac{px}{4\pi\varepsilon_0 r^3} \quad (\cos\vartheta = x/r)$$

(d)

$$E_x = -\frac{\partial V(r)}{\partial x} \to \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} + x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \frac{r^3 - 3x^2 r}{r^6}$$

$$E_x = -\frac{p}{4\pi\varepsilon_0}\frac{r^3 - 3x^2 r}{r^6} = \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos^2\vartheta - 1\right)$$

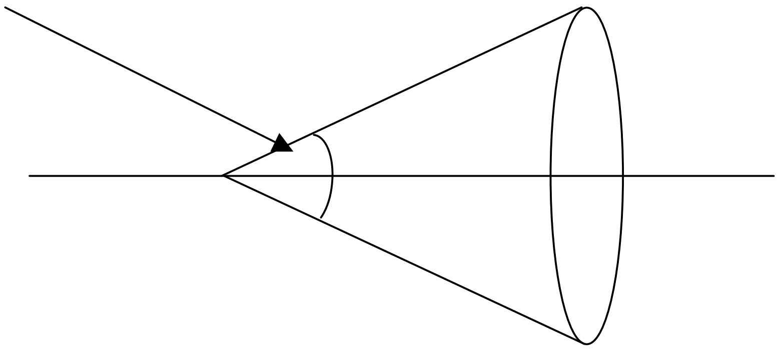
(e) $E_y = -\dfrac{\partial V}{\partial y} = -\dfrac{px}{4\pi\varepsilon_0}\left(-\dfrac{3}{r^4}\dfrac{\partial r}{\partial y}\right) = \dfrac{3pxy}{4\pi\varepsilon_0 r^5}$ since $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

$E_z = \dfrac{3pxz}{4\pi\varepsilon_0 r^5}$ similarly.

(f) Clearly $E_y = E_z = 0$ on the *y*-axis where $x = 0$.

(g) $E_x = 0$ when $3\cos^2\theta = 1$, i.e. when $\cos\theta = 1/\sqrt{3}$ and $\theta = \cos^{-1}(1/\sqrt{3}) = 54.7°$.

Vertex angle of cone $= 2\theta$



1st Year Electricity & Magnetism

Problem Sheet 2: ANSWERS

A. Net charge = +0.2 μC. So flux $= \dfrac{2\times10^{-7}}{\varepsilon_0} = 2.26\times10^{4}$ Vm.

B. Work out $\int_0^a 4\pi r^2 \rho dr$.

C. $\sigma = \varepsilon_0 E_{ext}$ so charge in area A of surface $= \varepsilon_0 E_{ext} A$.
Available charge in volume At of the plate (thickness t) $= NAte$ where N is the number density of free electrons. So the proportion on the surface $= \dfrac{\varepsilon_0 E_{ext}}{Nte} = 6.5\times10^{-13}$. Even if the plate thickness were 1 nm, there would therefore still be an ample supply!

D. See lecture notes.

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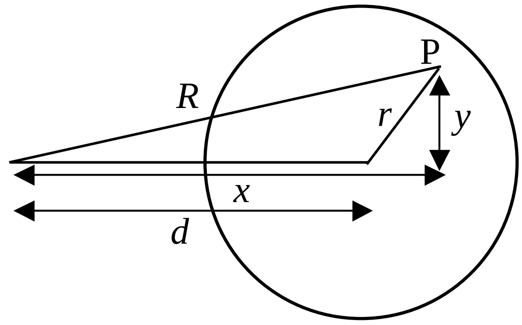
1. Consider the field at a general point P (x, y) within the hollow cavity; the origin is at the centre of the larger sphere.



Charge on the complete larger sphere _closer to the centre than_ P is $4\pi R^3 \rho/3$.

Charge on the complete _smaller_ sphere _closer to the centre than_ P is $4\pi r^3 \rho/3$.

Field at P (x-component) = $\dfrac{4\pi R^3\rho/3}{4\pi\varepsilon_0 R^2}\dfrac{x}{R} - \dfrac{4\pi r^3\rho/3}{4\pi\varepsilon_0 r^2}\dfrac{x-d}{r} = \dfrac{\rho d}{3\varepsilon_0}$.

Field at P (y-component) = $\dfrac{4\pi R^3\rho/3}{4\pi\varepsilon_0 R^2}\dfrac{y}{R} - \dfrac{4\pi r^3\rho/3}{4\pi\varepsilon_0 r^2}\dfrac{y}{r} = 0$.

Hence the field within the cavity is uniform in magnitude and direction.

2. $E_z = \displaystyle\int_0^a \frac{2\pi r\sigma\cos\alpha}{4\pi\varepsilon_0(r^2+z^2)}dr = \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{rz}{(r^2+z^2)^{3/2}}dr = \frac{\sigma z}{2\varepsilon_0}\left[\frac{-1}{(r^2+z^2)^{1/2}}\right]_0^a = \frac{\sigma z}{2\varepsilon_0}\left[\frac{-1}{(a^2+z^2)^{1/2}} + \frac{1}{z}\right]$

where $\cos\alpha = \dfrac{z}{\sqrt{r^2+z^2}}$ resolves the field in the x direction. It follows that

$$E_z = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right] = \frac{\sigma}{2\varepsilon_0}[1-\cos\alpha]$$

for $\alpha \to \pi/2$ the equation trivially returns the value for an infinite sheet. For $\alpha \to 0$ it seems to give the wrong answer. We can re-write the result as

$$E_z = \frac{Q}{2\pi\varepsilon_0 a^2}[1-\cos\alpha]$$

where $a = z\tan\alpha$ with z being the axial distance between disk and observation point. If this expression is substituted back to the formula for E_z and the L'Hospital rule is used, one obtains the equation for a point charge.

3. (a) Field at P due to dy element $= \dfrac{\lambda dy}{4\pi\varepsilon_0 r^2} = \dfrac{\lambda \cos^2\theta\, dy}{4\pi\varepsilon_0 x^2}$ since $x = r\cos\theta$.

(b) Multiply by a further $\cos\theta$ to resolve the field in the x-direction, and double the result to include the corresponding element at $-y$.

(c) $y = x\tan\theta$ so $dy = x\sec^2\theta\, d\theta$.

(d) Total field = $\displaystyle\int_0^{\pi/2} \frac{\lambda\cos\theta\, d\theta}{2\pi\varepsilon_0 x} = \frac{\lambda}{2\pi\varepsilon_0 x}\left[\sin\theta\right]_0^{\pi/2} = \frac{\lambda}{2\pi\varepsilon_0 x}$

(e) <u>*OR*</u> set $\cos\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$ to obtain

Total field

$$= \int_0^{\infty} \frac{\lambda x^3}{2\pi\varepsilon_0 x^2} \frac{dy}{(x^2 + y^2)^{3/2}} = \frac{\lambda x}{2\pi\varepsilon_0}\left[\frac{y}{x^2\sqrt{x^2 + y^2}}\right]_0^{\infty} = \frac{\lambda x}{2\pi\varepsilon_0}\left[\frac{1}{x^2} - 0\right] = \frac{\lambda}{2\pi\varepsilon_0 x}$$

as before.

(f) If you try to work out the potential, you get an infinite result. The potential is related to the work done in bringing a charge from infinity up to the wire, and this really <u>*is*</u> infinity for a wire of infinite length. All real wires are of course of finite length.

4. $C = \dfrac{\varepsilon_0 \varepsilon A}{s} = 0.018\mu\text{F}$.

When the sandwich is rolled up, the charge is spread over both surfaces of the conductors. This means the surface charge density, the field between the conductors, and the potential difference between them, are all halved. Hence, since $C = Q/V$, the capacitance doubles.

Note also that, if the field is halved, the stored energy per unit volume drops by a factor four. Since the volume between the conductors is doubled when the sandwich is rolled up, the stored energy U is therefore halved. But $U = \frac{1}{2}Q^2/C$ and Q is constant, so the conclusion once again is that C is doubled.

5. (i) $C = 4\pi\varepsilon_0 a$ so $U = \dfrac{Q^2}{2C} = \dfrac{q^2}{8\pi\varepsilon_0 a}$.

(ii) $E(r > 0) = \dfrac{q}{4\pi\varepsilon_0 r^2}$ so $dU = \frac{1}{2}\varepsilon_0 \dfrac{q^2}{(4\pi\varepsilon_0)^2 r^4} 4\pi r^2 dr$.

$$U = \frac{q^2}{4\pi\varepsilon_0}\int_a^{\infty}\frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0 a}$$

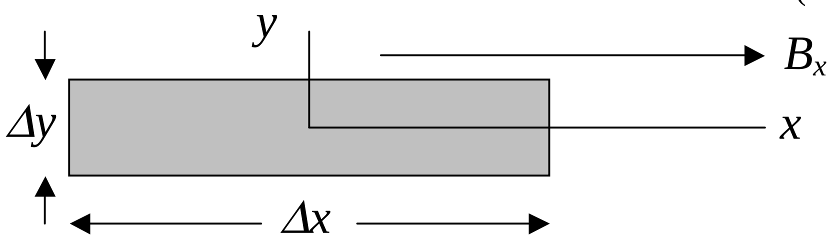
(iii) $\dfrac{e^2}{8\pi\varepsilon_0 R} = mc^2 \;\Rightarrow\; R = \dfrac{e^2}{8\pi\varepsilon_0 mc^2} = 1.4\times10^{-15}\text{m}$.

Problem Sheet 3: ANSWERS

A. $I = Ne\bar{v}A$ so $\bar{v} = \dfrac{I}{NeA} = \dfrac{1}{8.5\times10^{28}\ 1.6\times10^{-19}\ \pi\ (5\times10^{-4})^2} = 9.4\times10^{-5}$ m/s.

B. Resistance $R = L\rho / A$; Power $= I^2R$; Energy in 1 min $= \dfrac{100\ 1.7\times10^{-8}\ 60}{\pi\ (5\times10^{-4})^2}$ = 130 J.

C.



The Hall voltage is registered across the y-dimension.

$$V_{Hall} = \frac{IB\Delta y}{NeA} = \frac{IB}{Ne\Delta x} \text{ since } A = \Delta x \Delta y$$

$$V_{Hall} = \frac{3}{8.5\times10^{28}\ 1.6\times10^{-19}\ 1.5\times10^{-3}} = 0.15\ \mu\text{V}.$$

D. $f = \dfrac{\mu_0}{2\pi}\dfrac{I^2}{d} = \dfrac{4\pi\times10^{-7}\ 10^6}{2\pi\ 0.05} = 4\text{N/m}.$

E. $B = \dfrac{\mu_0 I}{2\pi d} = 4\times10^{-3}\text{T}.$

F. $F = \dfrac{\mu_0 IN}{2a}$ so $I = \dfrac{2aB}{\mu_0 N} = \dfrac{2\ 0.08\ 3\times10^{-3}}{4\pi\times10^{-7}\ 20} = 19.1A$

G. B points north. $I = \dfrac{2\pi rB}{\mu_0} = \dfrac{2\pi\ 20\times10^{-4}}{4\pi\times10^{-7}}$ =10 kA.

1. $$B = \frac{\mu_0 Ia}{4\pi}\int_{-L/2}^{+L/2}\frac{dy}{(a^2+y^2)^{3/2}} = \frac{\mu_0 Ia}{4\pi}\frac{1}{a^2}\left[\frac{y}{\sqrt{(a^2+y^2)}}\right]_{-L/2}^{+L/2} = \frac{\mu_0 I}{4\pi a}\frac{L}{\sqrt{(a^2+L^2/4)}} = \frac{\mu_0 I\cos\theta}{2\pi a}$$

$\rightarrow \dfrac{\mu_0 I}{2\pi a}$ as $L \rightarrow \infty$ and $\theta \rightarrow 0$,

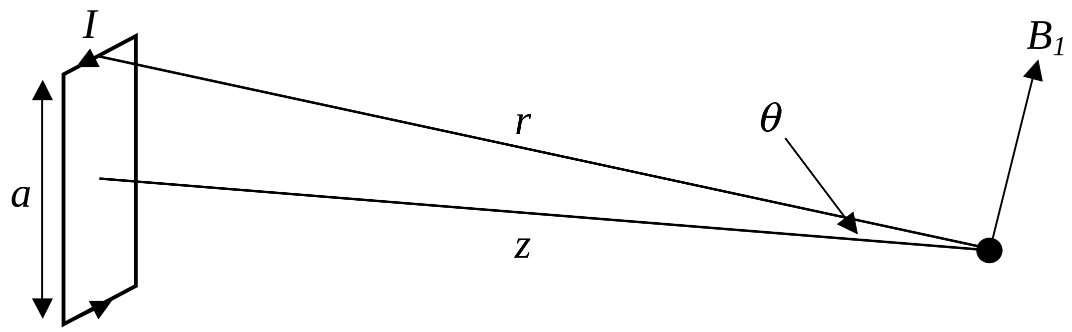
2. $B = \dfrac{\mu_0 I\cos 45^{\circ}}{2\pi}\left(\dfrac{1}{b} - \dfrac{1}{3b}\right) = \dfrac{4\pi\times10^{-7}\times3}{2\pi\sqrt{2}\ 4\times10^{-3}}\ \dfrac{2}{3} = 70.7\ \mu\text{T}$ directed into the paper.

3. Magnetic field from a ring of radius r, width dr, charge dq, moving with velocity $v = r\omega$ is

$$dB = \frac{\mu_0 dI}{2r} = \frac{\mu_0 v dq}{2r} = \frac{\mu_0 r\omega dq}{2r} = \frac{\mu_0 \omega\rho t dr}{2}$$

The field from the entire disk is therefore $B = \dfrac{\mu_0 a\rho t\omega}{2}$.

4.



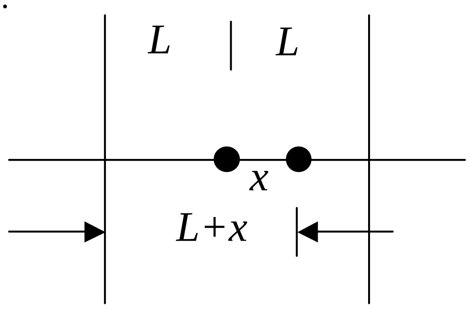
Magnetic field from top side of square $B_1 \cong \frac{\mu_0}{4\pi}\frac{Ia}{z^2}$ where $z \cong r$ for z ? a. Resolving this field along the z-direction and multiplying by 4 (because the square has four sides) yields $B_z = 4\frac{\mu_0 Ia}{4\pi z^2}\frac{a}{2z} = \frac{\mu_0 Ia^2}{2\pi z^3} = \frac{\mu_0 m}{2\pi z^3}$.

5. (i) $m = IA = I\pi a^2$. But $I = \frac{qv}{2\pi a}$ and $L = Mva$. So $m = \frac{qv\pi a^2}{2\pi a} = \frac{qL}{2m}$.

(ii) $m = \frac{e\hbar}{2m_e} = \frac{1.6\times10^{-19}\ 6.63\times10^{-34}}{4\pi\ 9.1\times10^{-31}} = 9.28\times10^{-24}$ A m^2.

(iii) $\Delta U = 2mB = 1.86\times10^{-22}$ J.

6.



$$B\,(\text{at P}) = \frac{\mu_0 NIa^2}{2}\left(\frac{1}{(a^2+(L+x)^2)^{3/2}} + \frac{1}{(a^2+(L-x)^2)^{3/2}}\right)$$

$$B_P = \frac{\mu_0 NIa^2}{2}(p_+^{-3/2} + p_-^{-3/2}) \text{ where } p_\pm = a^2 + (L\pm x)^2.$$

$$\frac{dB}{dt} = \frac{\mu_0 NIa^2}{2}\left(-\frac{3p_+^{-5/2}}{2}(2(L+x)) - \frac{3p_-^{-5/2}}{2}(-2(L-x))\right)$$

$$\frac{dB}{dt} = \frac{-3\mu_0 NIa^2}{2}((L+x)p_+^{-5/2} - (L-x)p_-^{-5/2}) = 0 \text{ since } p_+ = p_- \text{ at } x = 0\ .$$

Similar rather laborious analysis yields

$$\frac{d^2B}{dt^2} = \frac{-3\mu_0 NIa^2}{2}(p_+^{-7/2}(a^2 - 4(L+x)^2) + p_-^{-7/2}(a^2 - 4(L-x)^2))$$

This is zero at $x = 0$ when $a^2 = 4L^2$, i.e. when $2L = a$ and the coil separation is equal to the coil radius.

Turning charged insulator wheel:

The magnetic field on a ring of charge, radius r, rotating with angular frequency of ω is

$$dB = \frac{\mu_0 dI}{2r} = \frac{\mu_0 \omega dq}{2r}$$

but dq is the charge on the ring that has thickness t and width of dr so the volume is

$$V = 2\pi r \times dr \times t$$

Thus the charge is given as

$$dq = \rho 2\pi r dr t$$

↑ charge density

so

$$dB = \frac{\mu_0 \omega dq}{2r} = \frac{\mu_0 \omega \rho 2\pi r t dr}{2r} =$$

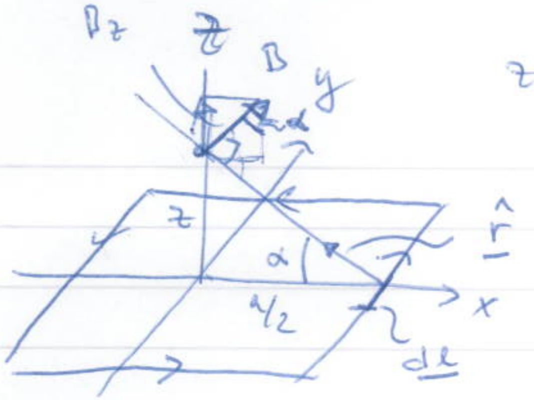
$$= \pi \mu_0 \omega \rho t dr$$

Thus for the whole disk

$$B = \int dB = \pi \mu_0 \omega \rho t a = \frac{\mu_0 \omega Q}{a}$$

as $\pi a^2 \times t$ = volume of disk

$$Q = \rho \times (\pi a^2 t)$$



$z \gg a$

$B_z = B \cdot \cos\alpha$

$$\underline{dB} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\left(\underline{dl} \times \hat{\underline{r}}\right)$$

$$\underline{B} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\left(\underline{a} \times \hat{\underline{r}}\right) \qquad |\underline{a} \times \hat{\underline{r}}| = a \cdot |r| \cdot \sin 90 = a$$

$$B_z = B \cdot \cos\alpha = B \cdot \frac{a}{2r}$$

$$B_z = \frac{\mu_0}{4\pi}\frac{I}{r^2}\, a \cdot \frac{a}{2r} = \frac{\mu_0 I a^2}{4\pi 2 r^3}$$

$$B_z \text{ total} = 4 \times B_z = \frac{\mu_0 I a^2}{2\pi r^3} = \frac{\mu_0 m}{2\pi z^3}$$

↑

$a \ll z$; $r \sim z$