PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 1

1. If $f(x) = x^2 - 3x + 2$, find $f(0), f(x^2), f(x+1)$. For what values of x does $f(x) = 0$? For what values of x does $f(2x) = 0$?

2. Find the inverse of each of the functions:

$$\text{(a)} \quad f(x) = 3x + 4, \quad \text{all real } x;$$

$$\text{(b)} \quad f(x) = 2x + x^2, \quad 0 < x < 1.$$

3. Are the following functions even, odd or neither?

$$\text{(a)} \quad x^2 + 2\sin x; \qquad \text{(b)} \quad (1+x^4)^{-1}\cos 3x;$$

$$\text{(c)} \quad x + |x|; \qquad \text{(d)} \quad \sin^3 x.$$

4. Evaluate the following limits:

$$\text{(a)} \quad \lim_{x\to\infty} \frac{x^2+1}{x^2-1}; \qquad \text{(b)} \quad \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

5. Evaluate the limits:

$$\text{(a)} \quad \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right); \qquad \text{(b)} \quad \lim_{x\to 1} \frac{x^9+x-2}{x^4+x-2}.$$

Hint for (b): Either use L'Hôpital's Rule or put $x = 1 + h$ and use the binomial expansion.

Starred Question

6* Given the definitions (from the lectures) of the hyperbolic functions

$$\cosh x = \frac{(e^x + e^{-x})}{2} \qquad \sinh x = \frac{(e^x - e^{-x})}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

show that

1. $\cosh^2 x - \sinh^2 x = 1$,
2. $\cosh^2 x + \sinh^2 x = \cosh 2x$,
3. $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \sinh x_2 \cosh x_1$,
4. $\frac{d}{dx}\tanh x = \text{sech}^2 x \qquad (\text{sech}\, x = \frac{1}{\cosh x})$.

Note the differences in the signs in 1) and 2) from the trigonometric cases.

PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 2

1. Differentiate $x^3 \cos(5x+1)$; $\ln(\sec x + \tan x)$; $x/(x+1)$.

2. Find dy/dx when (a) $y^3 = x^3 - xy$; (b) $xe^y = \cos(xy)$.

3. Sketch the graphs of the following:
   (a) $y = x + 1/x$, $(x \neq 0)$;
   (b) $y = \ln(1-x^2)$, $-1 < x < 1$;
   (c) $r = a(1 - \cos\theta)$ where r and θ are polar coordinates and a is a positive constant.

   Note: *Plane polar co-ordinates (r, θ) are related to Cartesian co-ordinates (x, y) by $x = r\cos\theta$ and $y = r\sin\theta$: hence $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.*

4. Find the stationary points of the function $f(x) = x^2(1-x)^3$ and determine their nature. Sketch the graph $y = f(x)$.

5. If $r(1+\cos\theta) = 2$, where r and θ are plane polar coordinates, express the equation in terms of cartesian coordinates (x, y); show that the graph is a parabola and sketch it.

STARRED PROBLEMS

6* Differentiate $y = \sin^{-1}\{x/(1+x)\}$ and $y = \sec^{-1}(x)$.

7* Find where the function

$$f(x) = \frac{2x^2 - 5x - 25}{x^2 + x - 2}$$

is discontinuous. Find also the points where it is zero, its limiting values as $x \to \pm\infty$ and its maximum and minima. Hence sketch its graph.

PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 3

1. Integrate by parts:

$$\text{(a)} \quad x^3 \sin x; \qquad \text{(b)} \quad \tan^{-1} x.$$

2. Evaluate the integrals:

$$\text{(a)} \int_0^1 (1+x^2)^{-3/2} dx \qquad \text{(b)} \int_0^\infty \left(1+e^{2x}\right)^{-1} dx, \qquad \text{(c)} \int_1^{3/2} (2-x)^{-1}(x-1)^{-1/2} dx.$$

Hint: In (a) substitute $x = \tan\theta$; in (b) substitute $u = e^{2x}$; in (c) use the substitution $(x-1) = u^2$.

3. Which of the following integrals are convergent?

$$\text{(a)} \quad \int_0^1 \ln x \, dx \; ; \qquad \text{(b)} \quad \int_0^1 (x-1)^{-2} \, dx \; ;$$

4. Show that

$$\int x^k \ln x \, dx = \frac{x^{k+1}}{(k+1)^2} \left[(k+1)\ln x - 1\right] + c$$

where c is a constant and $k \neq -1$.

STARRED PROBLEMS

5* Which of the following integrals are convergent?

$$\text{(a)} \quad \int_1^\infty \ln x \, dx \; ; \qquad \text{(b)} \quad \int_0^\infty e^{-ax} \sin bx \, dx, \quad (a > 0).$$

6* Integrate

$$\text{(a)} \quad \frac{x^4}{x^2+1} \; ; \qquad \text{(b)} \quad \frac{1}{x \ln x}.$$

PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 4

1. Put into partial fractions and hence find the indefinite integral of

$$f(x) = \frac{2x^2 - x + 2}{x(x-1)(x+1)}.$$

2. By using the trigonometric formula $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$ calculate the indefinite integral

$$I = \int \sin 3x \cos 5x \, dx.$$

3. Recall from the lectures that the mean value $\overline{f}$ of a function $f(x)$ over an interval $0 \leq x \leq a$ is given by

$$\overline{f} = \frac{\int_0^a f(x)\, dx}{\int_0^a dx}.$$

Find the mean value of $f(x) = \sin x$ in the interval $0 \leq x \leq \pi$, and of $f(x) = \sin^2 x$ in the interval $0 \leq x \leq 2\pi$.

4. If

$$I_n = \int_0^{\pi/2} \sin^n x \, dx,$$

where $n \geq 0$ is an integer, show that $I_n = \frac{n-1}{n} I_{n-2}$, for $n \geq 2$. Hence show that

$$I_8 = \int_0^{\pi/2} \sin^8 x \, dx = \frac{35}{256}\pi.$$

STARRED PROBLEMS

5* Calculate the length of the curve

$$y = \frac{x^3}{a^2} + \frac{a^2}{12x},$$

from $x = a/2$ to $x = a$, where a is a positive constant.

6* If

$$I = \int_0^{\pi/2} \frac{\sin^{1/3} x}{\sin^{1/3} x + \cos^{1/3} x} \, dx,$$

use the substitution $x = \pi/2 - y$ to show that

$$I = \int_0^{\pi/2} \frac{\cos^{1/3} x}{\sin^{1/3} x + \cos^{1/3} x} \, dx.$$

Hence show that $I = \pi/4$.

PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 5

Recall from your notes that:

(i) Plane polar co-ordinates (r,θ) are related to Cartesian co-ordinates (x,y) by $x = r\cos\theta$ and $y = r\sin\theta$: hence $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
(ii) In Cartesian co-ordinates a small element of arc length ds is related to the small elements dx and dy by $(ds)^2 = (dx)^2 + (dy)^2$ (with an additional $(dz)^2$ in 3D). In plane polar co-ordinates this converts to $(ds)^2 = (dr)^2 + r^2(d\theta)^2$.
(iii) Volume of revolution is $\pi\int_a^b y^2\,dx$ whose surface area is $2\pi\int_a^b y\,ds$.

1. Find the lengths of the following curves:

    (a) The catenary $y = \cosh x$ from $x = 0$ to $x = 1$. [Answer: $\sinh 1$].

    (b) The circular helix expressed in parametric form $x = \cos t$, $y = \sin t$ and $z = t$ from $t = 0$ to $t = 2\pi$. [Answer: $2\sqrt{2}\pi$].

    (c) The curve $y = x^{3/2}$ from $(0,0)$ to $(4,8)$. [Answer: $\frac{8}{27}\left(10^{3/2} - 1\right)$].

    (d) One branch of the 4-cusped hypocycloid expressed in parametric form $x = \cos^3 t$, $y = \sin^3 t$ from $t = 0$ to $t = \pi/2$. [Answer: $3/2$].

2. Show that the area of one loop $(-\pi/4 \leq \theta \leq \pi/4)$ of the lemniscate $r^2 = a^2\cos 2\theta$ is $a^2/2$.

3. Find the position of the centre of mass of a uniform thin wire in the form of a circular arc of radius a, subtending an angle of 2γ at the centre. [Answer: $a\sin\gamma/\gamma$ from the apex.]

4. Find the area enclosed by the ellipse $(x/a)^2 + (y/b)^2 = 1$. Assuming this elliptical area to be of uniform density, find also the position of the centre of gravity of the part that lies in the first quadrant. [Answers: πab and $(4a/3\pi, 4b/3\pi)$.]

STARRED PROBLEMS

5* Show that $8a$ is the total length of the closed curve called the cardioid (heart shape)

$$r = a(1 - \cos\theta).$$

6* Show that the length of one arch $(0 \leq t \leq 2\pi)$ of the cycloid defined by

$$x = a(t - \sin t) \qquad y = a(1 - \cos t)$$

is $8a$. Show that the area of the surface obtained by a complete revolution of this arch about the x-axis is $64\pi a^2/3$.

PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 6

1. Calculate $\partial u/\partial x$ and $\partial u/\partial y$ if $u = 4x^2y - y^2 + 3x - 1$.

2. Find the relation between the constants α and β if the function $u = e^{\alpha x}\cos\beta y$ satisfies Laplace's equation:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$
Show also that the following function $u(x, y)$ satisfies Laplace's equation
$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.$$

3. If $g = \tan^{-1}(y/x)$, calculate $\partial g/\partial x$ and $\partial g/\partial y$, and show that
$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 0$$

4. If $u = x^2 + 3y^3$ and $x = s + t, y = 2s - t$, calculate $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ (i) by using the chain rule, and (ii) by first expressing u as a function of s and t.

5. A closed box has variable sides of length x, y and z but a fixed volume V. Show that the shape of the box is a cube when the surface area A is minimum. *Note:* at a stationary point of a function of two variables $a = a(x, y)$ the two partial derivatives a_x and a_y need to be zero simultaneously.

STARRED PROBLEMS

6* If $u = x\ln(x^2 + y^2) - 2y\tan^{-1}(y/x)$, verify that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + 2x.$$

7* The equation of state[1] of a gas, relating pressure p, volume V and temperature T, is $f(p, V, T) = 0$ and hence
$$df = \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial V}dV + \frac{\partial f}{\partial T}dT = 0.$$
Show that
$$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{(\partial f/\partial V)_{p,T}}{(\partial f/\partial p)_{V,T}}$$
and obtain similar expressions for $(\partial V/\partial T)_p$ and $(\partial T/\partial p)_V$. Deduce that
$$\left(\frac{\partial p}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p\left(\frac{\partial T}{\partial p}\right)_V = -1.$$

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[1] In this problem we need not specify the function f; it is left as an arbitrary function of the three independent variables p, V and T but for an ideal gas it would take the form $f = pV - RT = 0$. In fact, you can verify some of the above formulae using this relation.

PHYSICS 1: MATHEMATICAL ANALYSIS I.

PROBLEMS 7

1. If f and g are any twice-differentiable functions, use the chain rule, along with the new variables $s = x + y$ and $t = x + \frac{1}{2}y$, to show that
$$V(x, y) = f(x + y) + g(x + \tfrac{1}{2}y)$$
satisfies the partial differential equation
$$V_{xx} - 3V_{xy} + 2V_{yy} = 0,$$
where the suffices denote partial derivatives.

2. If $u = u(x, y)$ and x and y transform into two new variables s and t such that $s = \frac{x}{x^2+y^2}$ and $t = \frac{y}{x^2+y^2}$, show that
$$u_s^2 + u_t^2 = \left(u_x^2 + u_y^2\right)\left(x^2 + y^2\right)^2.$$

3. Are the following exact differentials? If so, of what functions?

   (i) $e^y dx + x(e^y + 1)dy$; (ii) $(e^y + ye^x)dx + (e^x + xe^y + 1)dy$

STARRED QUESTION

4* If $u = u(x, y)$ and x and y are related to two new independent variables s and t by
$$x = st, \qquad y = \frac{s+t}{s-t},$$
use the chain rule to find $\frac{\partial u}{\partial s}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Solve this to show that
$$2x\frac{\partial u}{\partial x} = s\frac{\partial u}{\partial s} + t\frac{\partial u}{\partial t},$$
and
$$4y\frac{\partial u}{\partial y} = \left(s^2 - t^2\right)\left(\frac{1}{s}\frac{\partial u}{\partial t} - \frac{1}{t}\frac{\partial u}{\partial s}\right).$$

PHYSICS 1: MATHEMATICAL ANALYSIS I.

PROBLEMS 8

1. Use the ratio test to determine whether the following two series are convergent

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{(3-4i)^n}{n!}.$$

2. Show that $y = \tan x$ satisfies the equation

$$\frac{dy}{dx} = 1 + y^2.$$

By repeated differentiation of this result, find the higher derivatives that are required to determine the first three non-zero terms of the Maclaurin series for $\tan x$.

3. Show that there are two stationary values of the function

$$u(x, y) = \frac{x^2 + y^2 + 2x + 1}{x + y}.$$

By considering the second partial derivatives u_{xx}, u_{yy} and u_{xy}, show that one is a maximum and the other is a minimum.

4. Sketch contours (curves of constant u) for the function $u = xy(x+y-1)$ and indicate regions where u is zero, positive and negative respectively. Locate the stationary points of the function and deduce their nature from the contour diagram. Now use the standard method of calculating the sign of $\left(u_{xy}^2 - u_{xx}u_{yy}\right)$ etc at each stationary point to confirm your findings.

STARRED PROBLEM

5* Show that the function

$$u(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$$

has three stationary points, two of which are minima, the other being a saddle.

MATHEMATICAL ANALYSIS SOLUTIONS 1. M.Phys I.

1) $f(0) = 2$; $f(x^2) = x^4 - 3x^2 + 2$

$f(x+1) = (x+1)^2 - 3(x+1) + 2 = x^2 - x$

$f(x) = 0$ at roots of $x^2 - 3x + 2 = 0$, namely $x = 1$ and 2

so $f(2x) = 0$ at $x = \frac{1}{2}$ and 1.

2) a) $y = 3x + 4 \Rightarrow x = \frac{1}{3}(y-4)$

Hence $f^{-1}(x) = \frac{1}{3}(x-4)$ for all x.

b) $y = 2x + x^2$ $(0 < x < 1)$ so the range is $0 < y < 3$.

Solve the quadratic in x i.e. $x^2 + 2x - y = 0$,

which gives $x = -1 \pm (1+y)^{1/2}$. Note that the

2nd root $x = -1 - (1+y)^{1/2}$ is <u>negative</u> and out

of the domain $0 < x < 1$. We reject this root,

leaving $f^{-1}(x) = -1 + (1+x)^{1/2}$ for $0 < x < 3$

3) (a) Neither (b) Even (c) Neither (d) Odd

4) (a) $\lim\limits_{x \to \infty} \frac{x^2+1}{x^2-1} = \lim\limits_{x \to \infty} \frac{1 + 1/x^2}{1 - 1/x^2} = 1$

(b) $\lim\limits_{x \to 0} \frac{(1+x)^{1/2} - (1-x)^{1/2}}{x} = \lim\limits_{x \to 0} \frac{1}{x}\left[(1 + \frac{1}{2}x + \ldots) - (1 - \frac{1}{2}x \ldots)\right]$ Binomial Thm.

$= 1$

5) (a) $\lim\limits_{x \to \infty} x \sin\left(\frac{1}{x}\right) = \lim\limits_{y \to 0} \left(\frac{\sin y}{y}\right) = 1$ $(y = 1/x)$

(b) $\lim\limits_{x \to 1} \frac{x^9 + x - 2}{x^4 + x - 2} = \lim\limits_{h \to 0} \frac{(1+h)^9 + (1+h) - 2}{(1+h)^4 + (1+h) - 2}$

$= \lim\limits_{h \to 0} \frac{(1 + 9h + \ldots) + (1+h) - 2}{(1 + 4h + \ldots) + (1+h) - 2} = \frac{10}{5} = 2$

Alternatively, L'H Rule gives,

$\lim\limits_{x \to 1} \frac{x^9 + x - 2}{x^4 + x - 2} = \lim\limits_{x \to 1} \frac{9x^8 + 1}{4x^3 + 1} = \frac{10}{5} = 2.$

MPhys I

Mathematical Analysis

Solutions to Sheet 2

(1) (i) $\frac{dy}{dx} = 3x^2\cos(5x+1) - 5x^3\sin(5x+1)$ (Product Rule)

(ii) $\frac{dy}{dx} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x$ (Note $\frac{d}{dx}\sec x = \frac{\sin x}{\cos^2 x} = \sec x \tan x$)

(iii) $\frac{dy}{dx} = \frac{x+1-x}{(x+1)^2} = (x+1)^{-2}$ (Quotient Rule)

(2) (a) $y^3 = x^3 - xy$ so $3y^2\frac{dy}{dx} = 3x^2 - (y + x\frac{dy}{dx})$

Hence $\frac{dy}{dx} = \frac{3x^2-y}{3y^2+x}$

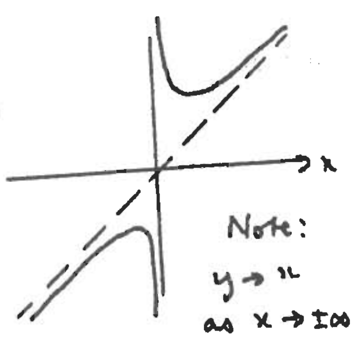
(b) $xe^y = \cos xy$: LHS : $\frac{d}{dx}(xe^y) = e^y + x\frac{dy}{dx}e^y$

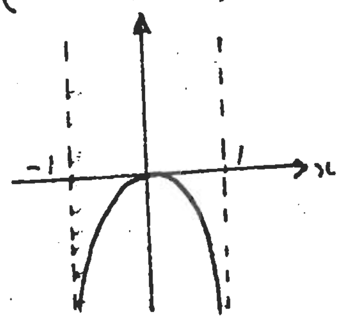
RHS : $\frac{d}{dx}(\cos xy) = -\sin(xy)(y + x\frac{dy}{dx})$

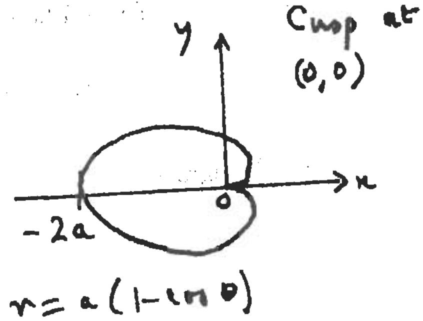
$\therefore \frac{dy}{dx}(xe^y + x\sin xy) = -(y\sin xy + e^y)$

so $\frac{dy}{dx} = -\frac{y\sin xy + e^y}{x(e^y + \sin xy)}$

(3) (a)

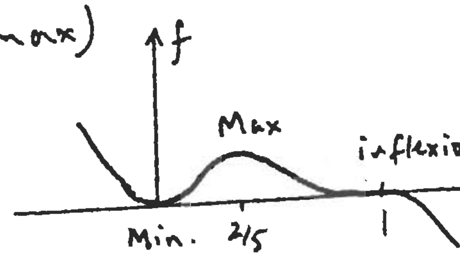






(4) Stationary pts when $f' = 0$: $f'(x) = 2x(1-x)^3 - 3x^2(1-x)^2$

$x = 0$ (min), $x = 1$ (inflexion), $x = 2/5$ (max)

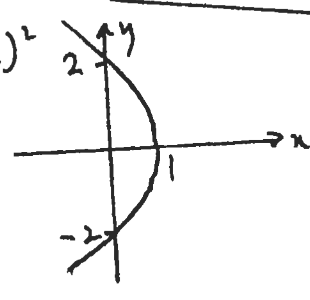


(5) $r(1+\cos\theta) = 2 \Rightarrow r + x = 2$

because $x = r\cos\theta$. Hence

$r^2 = (2-x)^2$ so $x^2 + y^2 = (2-x)^2$

$\therefore y^2 = 4(1-x)$ (Parabola)



M.Phys I

Mathematical Analysis

Solutions to Problems 3.

① a) $\int x^3 \sin x \, dx = -\int x^3 \, d(\cos x) = -x^3 \cos x + 3\int x^2 \cos x \, dx$

$= -x^3 \cos x + 3\int x^2 \, d(\sin x)$

$= -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x \, dx$

$= -x^3 \cos x + 3x^2 \sin x + 6\int x \, d(\cos x)$

$= -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6 \sin x + c$

b) $\int \underbrace{\tan^{-1}x}_{u} \, \underbrace{dx}_{v} = x \tan^{-1}x - \int x \frac{d}{dx}(\tan^{-1}x)\, dx$

$= x \tan^{-1}x - \int \frac{x\,dx}{1+x^2} = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c.$

② a) Use $x = \tan\theta$ $\quad \int_0^1 \frac{dx}{(1+x^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{\sec^3\theta} = \int_0^{\pi/4} \cos\theta \, d\theta = \frac{1}{\sqrt{2}}$

b) $\int_0^\infty \frac{dx}{1+e^{2x}} = \frac{1}{2}\int_1^\infty \frac{du}{u(1+u)} = \frac{1}{2}\int_1^\infty \left(\frac{1}{u} - \frac{1}{1+u}\right) du = \frac{1}{2}\left[\ln\left|\frac{u}{1+u}\right|\right]_1^\infty$

$= \frac{1}{2}\left[\ln 1 - \ln\frac{1}{2}\right] = \frac{1}{2}\ln 2.$

c) $\int_1^{3/2} \frac{dx}{(2-x)(x-1)^{1/2}} = \int_0^{1/\sqrt{2}} \frac{2\cancel{u}\,du}{(1-u^2)\cancel{u}} = \int_0^{1/\sqrt{2}} \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du$

$= \left[\ln\left|\frac{1+u}{1-u}\right|\right]_0^{1/\sqrt{2}} = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\ln(\sqrt{2}+1).$

③ a) $\int_\varepsilon^1 \ln x \, dx = \left[x(\ln x - 1)\right]_\varepsilon^1 = -1 - \varepsilon(\ln\varepsilon - 1) = \varepsilon - 1 - \varepsilon\ln\varepsilon$

What does $\varepsilon\ln\varepsilon$ do as $\varepsilon \to 0$? Actually $\lim_{\varepsilon\to 0}(\varepsilon\ln\varepsilon) \to 0$, so RHS $\to -1$ as $\varepsilon \to 0$. $\therefore$ Convergent.

b) $\int_0^{1-\varepsilon} \frac{dx}{(x-1)^2} = -\left[(x-1)^{-1}\right]_0^{1-\varepsilon} = \frac{1}{\varepsilon} - 1.$ Not convergent as $\varepsilon \to 0$.

④ $\int x^k \ln x \, dx = \frac{1}{k+1}\int \ln x \, d(x^{k+1}) = \frac{1}{k+1}\left[(\ln x)x^{k+1} - \int x^{k+1}x^{-1}dx\right]$

$= \frac{1}{k+1}\left[x^{k+1}\ln x - \frac{1}{k+1}x^{k+1}\right] + c \qquad k \neq -1.$

$= (k+1)^{-2}\left[(k+1)\ln x - 1\right]x^{k+1} + c.$

M Phys 1. Mathematical Analysis

SOLUTIONS TO PROBLEMS 4

(1) $f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \Rightarrow A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 2x^2 - x + 2.$

$(A, B, C) = (-2, 3/2, 5/2)$. Hence

$$\int f(x)\,dx = -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + c.$$

(2) $\sin 3x \cos 5x = \frac{1}{2}[\sin 8x - \sin 2x]$ from trig. formula

$\therefore \quad I = \frac{1}{2}\int(\sin 8x - \sin 2x)\,dx = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + c$

(3) $\overline{\sin x} = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}[\cos x]_0^{\pi} = -\frac{1}{\pi}[-1-1] = 2/\pi$

$\overline{\sin^2 x} = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x = \frac{1}{4\pi}\int_0^{2\pi}(1-\cos 2x)\,dx = \frac{1}{2}$

(4) $I_n = \int_0^{\pi/2} \sin^n x\,dx = -\int_0^{\pi/2} \sin^{n-1}x\,d(\cos x)$

$= -[\sin^{n-1}x \cos x]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}x \cos^2 x\,dx \qquad n \geq 2$

$= 0 + (n-1)\int_0^{\pi/2} \sin^{n-2}x\,(1-\sin^2 x)\,dx$

$= (n-1)[I_{n-2} - I_n]$

Solve for I_n: $\quad nI_n = (n-1)I_{n-2} \qquad n \geq 2$

$I_8 = \frac{7}{8}I_6 = \frac{7}{8}\cdot\frac{5}{6}I_4 = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}I_2 = \frac{7.5.3.1}{8.6.4.2}I_0$

$I_0 = \int_0^{\pi/2} dx = \pi/2$

$\therefore \quad I_8 = \frac{35}{256}\pi.$

Physics I Mathematical Analysis SOLUTIONS 5.

1) a) $y' = \sinh x$ $\quad s = \int_0^1 (1+\sinh^2 x)^{1/2}\,dx = \int_0^1 \cosh x\,dx = \sinh 1.$

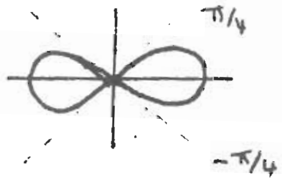
b) $x = \cos t,\ y = \sin t,\ z = t.$ $\quad (ds)^2 = [(-\sin t\,dt)^2 + (\cos t\,dt)^2 + (dt)^2]$

$$s = \int ds = \sqrt{2}\int_0^{2\pi} dt = 2\sqrt{2}\,\pi.$$

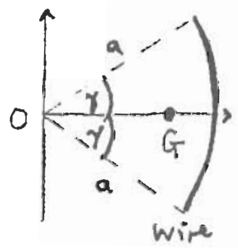
c) $y = x^{3/2}$ $\quad s = \int_0^4 \left[1 + \tfrac{9}{4}x\right]^{1/2} dx = \tfrac{8}{27}\left(10^{3/2} - 1\right)$

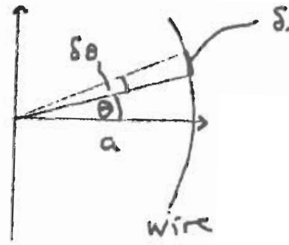
d) $x = \cos^3 t,\ y = \sin^3 t$ $\quad s = 9^{1/2}\int_0^{\pi/2}\left[\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right]^{1/2} dt$

$$\therefore\ s = 3\int_0^{\pi/2}\cos t\sin t\,dt = \frac{3}{2}\int_0^{\pi/2}\sin 2t\,dt = 3/2.$$

2) 

$$\text{Area} = \tfrac{1}{2}\int r^2 d\theta = \frac{a^2}{2}\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta = \tfrac{1}{2}a^2$$

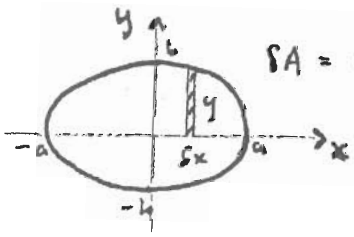
3) 



$$OG = \frac{\int x\,\rho\,ds}{\int \rho\,ds}$$

ρ is mass of wire/unit length.

$$= \frac{\rho\int_{-\gamma}^{\gamma} a^2\cos\theta\,d\theta}{\rho\int_{-\gamma}^{\gamma} a\,d\theta} = \frac{a\sin\gamma}{\gamma}$$

4) 

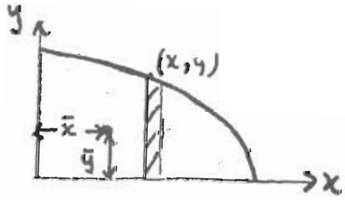
$\delta A = y\,\delta x$

$$\text{Area of ellipse} = 4\int_0^a y\,dx$$

$$= 4b\int_0^a \left(1 - \frac{x^2}{a^2}\right)^{1/2} dx$$

Put $x = a\cos\theta$ so $A = -4ab\int_{\pi/2}^{0}\sin^2\theta\,d\theta$ $\qquad \cos 2\theta = 1 - 2\sin^2\theta$

$$= \frac{4ab}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta = \pi ab.$$



A_1 area of 1st quadrant
ρ is mass/unit area.

$$\therefore\ \rho A_1\bar{x} = \int x\rho y\,dx$$

Mass of strip
$\delta M = \rho\,y\,\delta x$

$$\therefore\ A_1\bar{x} = \int_0^a x\,b\left(1 - x^2/a^2\right)^{1/2} dx$$

$$\therefore\ A_1\bar{x} = -a^2 b\int_{\pi/2}^{0}\cos\theta\sin^2\theta\,d\theta = \tfrac{1}{3}a^2 b$$

$A_1 = \frac{\pi ab}{4}$, the area of the 1st quadrant. Hence

$\bar{x} = \frac{4a}{3\pi}$. By symmetry $\bar{y} = \frac{4b}{3\pi}$.

1) $\frac{\partial u}{\partial x} = 8xy + 3 \qquad \frac{\partial u}{\partial y} = 4x^2 - 2y$

2) i) $u = e^{\alpha x}\cos\beta y \qquad \frac{\partial u}{\partial x} = \alpha e^{\alpha x}\cos\beta y, \quad \frac{\partial^2 u}{\partial x^2} = \alpha^2 e^{\alpha x}\cos\beta y$

$\frac{\partial u}{\partial y} = -\beta e^{\alpha x}\sin\beta y, \quad \frac{\partial^2 u}{\partial y^2} = -\beta^2 e^{\alpha x}\cos\beta y$

Need $\alpha^2 - \beta^2 = 0$ to satisfy Laplace's eqn. $\Rightarrow \alpha = \pm\beta$.

ii) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x, \quad \frac{\partial^2 u}{\partial x^2} = 6x + 6$

$\frac{\partial u}{\partial y} = -6xy - 6y \qquad \frac{\partial^2 u}{\partial y^2} = -6x - 6$

Sum is zero.

3) $g = \tan^{-1}(y/x) \qquad \therefore \frac{\partial g}{\partial x} = \frac{\frac{\partial}{\partial x}(y/x)}{1 + (y/x)^2} = \frac{-y/x^2}{1 + y^2/x^2} = \frac{-y}{x^2+y^2}$

$\frac{\partial g}{\partial y} = \frac{\frac{\partial}{\partial y}(y/x)}{1+(y/x)^2} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}$. Hence $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 0$

4) i) Chain Rule: $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$

$x = s+t \quad y = 2s - t \quad \frac{\partial x}{\partial s} = 1 \quad \frac{\partial x}{\partial t} = 1 \quad \frac{\partial y}{\partial s} = 2 \quad \frac{\partial y}{\partial t} = -1$

$= \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}$

$= 2x + 2.(9y^2)$

$= 2(s+t) + 18(2s-t)^2$

$= 18(4s^2 - 4st + t^2) + 2s + 2t \qquad -(*)$

Now $u = x^2 + 3y^3 = (s+t)^2 + 3(2s-t)^3 = s^2 + 2st + t^2 + 3(8s^3 - 12s^2t + 6st^2 - t^3)$

so $\frac{\partial u}{\partial s} = 2s + 2t + 3(24s^2 - 24st + 6t^2)$ Same as $(*)$

Do the same for $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$ etc.

5) $V = xyz$ (V fixed) $\quad A = 2(xy + yz + xz)$

Eliminate z in A using $z = V/xy$.

$\therefore \quad A = 2(xy + \frac{V}{x} + \frac{V}{y}) \qquad V$ constant.

$\therefore \quad \frac{\partial A}{\partial x} = 2\left(y - \frac{V}{x^2}\right), \quad \frac{\partial A}{\partial y} = 2\left(x - \frac{V}{y^2}\right)$

For $A_x = 0$ & $A_y = 0$ together we have

$V = x^2 y$ & $V = xy^2$ with $V = xyz$. Only solution is $x = y = z = V^{1/3}$. Minimum by inspection.

M. Phys 1. Mathematical Analysis

Solutions to Sheet 7.

1) $V(x,y) = f(x+y) + g(x+\tfrac{1}{2}y)$

Let $u = x+y$ $\quad v = x+\tfrac{1}{2}y$ $\qquad u_x = 1 \quad u_y = 1$, $\quad v_x = 1 \quad v_y = \tfrac{1}{2}$

so $V = f(u) + g(v)$

$$\therefore \frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial V}{\partial u} + \frac{\partial V}{\partial v} = f' + g' \qquad 1)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial V}{\partial u} + \frac{1}{2}\frac{\partial V}{\partial v} = f' + \tfrac{1}{2}g' \qquad 2)$$

Note that 1) implies that the derivative operation $\partial/\partial x$ can be written as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$ and similarly $$\frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}$$

$$\left.\begin{aligned}
\therefore \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(f'+g') = f'' + g'' \\
\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) &= \left(\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}\right)(f'+\tfrac{1}{2}g') = f'' + \tfrac{1}{4}g'' \\
\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(f'+\tfrac{1}{2}g') = f'' + \tfrac{1}{2}g''
\end{aligned}\right\}$$

Hence $V_{xx} + 2V_{yy} = 3V_{xy}$

2) $s = \dfrac{x}{x^2+y^2}$ $\qquad t = \dfrac{y}{x^2+y^2}$

$$\frac{\partial s}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad \frac{\partial s}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial t}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial t}{\partial y} = \frac{-y^2+x^2}{(x^2+y^2)^2}$$

Chain rule:

$$u_x = u_s\frac{\partial s}{\partial x} + u_t\frac{\partial t}{\partial x} = [u_s(y^2-x^2) - 2xy\,u_t](x^2+y^2)^{-2}$$

$$u_y = u_s\frac{\partial s}{\partial y} + u_t\frac{\partial t}{\partial y} = [-2xy\,u_s + (x^2-y^2)u_t](x^2+y^2)^{-2}$$

$$\therefore (u_x^2 + u_y^2)(x^2+y^2)^4 = u_s^2(x^2+y^2)^2 + u_t^2(x^2+y^2)^2$$ Hence result.

3) Write the differential as $P\,dx + Q\,dy$.

To be able to write this as df and find $f(x,y)$ we need (notes) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

i) $P = e^y$, $Q = x(e^y+1)$ $\quad$ Clearly $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ $\quad$ No!

ii) $P = e^y + ye^x$, $Q = e^x + e^y + 1$

$\dfrac{\partial P}{\partial y} = e^y + e^x$, $\quad \dfrac{\partial Q}{\partial x} = e^x + e^y$. Yes! $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$\underline{f = xe^y + y(e^x+1)}$$

M. Phys I Mathematical Analysis 3

Solutions to Sheet 8.

a) $u_n = \dfrac{1}{(n+1)!}$ $\therefore$ $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\dfrac{(n+1)!}{(n+2)!} = \lim_{n\to\infty}\dfrac{1}{n+2} = 0$. Convergent.

b) $u_n = \dfrac{(3-4i)^n}{n!}$ $\therefore$ $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\dfrac{n!}{(n+1)!}\left|\dfrac{(3-4i)^{n+1}}{(3-4i)^n}\right| = \lim_{n\to\infty}\dfrac{5}{n+1} = 0$. Convergent.

$\dfrac{dy}{dx} = \sec^2 x = 1+\tan^2 x$. Since $\tan x$ is odd, we need to go to x^5.

Differentiate the diff. equation 4 times: $y'' = 2yy'$; $y''' = 2(yy''+y'^2)$

$y^{(4)} = 2(yy''' + 3y'y'')$, $y^{(5)} = 2(yy^{(4)} + 4y'y''' + 3y''^2)$.

At $x=0$: $y=0$, $y'=1$, $y''=0$, $y'''=2$, $y^{(4)}=0$, $y^{(5)}=16$.

$$\therefore\quad y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \frac{x^5}{5!}y^{(5)}(0)\ldots$$

$$= 0 + x + 0 + \frac{2}{3!}x^3 + 0 + \frac{16}{5!}x^5\ldots$$

$$\therefore\quad y = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

3) $u_x = \dfrac{x^2+2xy+2y-y^2-1}{(x+y)^2}$; $u_y = \dfrac{2xy+y^2-x^2-2x-1}{(x+y)^2}$

$u_x = u_y = 0$ Add to get $2xy = x-y+1$. Subtract to get $x-y+1=0$ $(x+y\neq 0)$.

Together we have $x=0, y=1$ and $y=0, x=-1$. Two points $(0,1), (-1,0)$.

After a lot of work: $u_{xx} = \dfrac{4y^2-4y+2}{(x+y)^3}$ $u_{yy} = \dfrac{4x^2+4x+2}{(x+y)^3}$

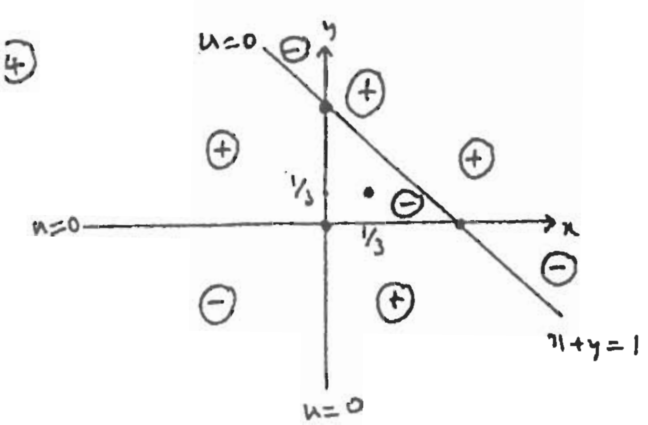
$u_{xy} = \dfrac{2(x-y-2xy+1)}{(x+y)^3}$

$(-1,0)$: $u_{xx}=-2$, $u_{yy}=-2$, $u_{xy}=0$ MAX.

$(0,1)$ $u_{xx}=2$, $u_{yy}=2$, $u_{xy}=0$ MIN.

(Note: At the max $u=0$ while at the min $u=2$. How can this be? Consider that u becomes infinite along the line $y=-x$.)

4)



Signs in circles refer to the sign of u.

$u_x = y(2x+y-1)$, $u_y = x(x+2y-1)$

$\therefore$ Stat. pts at $(0,0), (0,1), (1,0)$ & $(\frac{1}{3},\frac{1}{3})$.

Consider changes of sign in u across each point. Clearly $(0,0), (0,1), (1,0)$ are SADDLES.

Check $u_{xy}^2 - u_{xx}u_{yy} > 0$. This is 1 for these points, and is $-1/3$ for $(1/3, 1/3)$. This is a MINIMUM as $u_{xx}>0$ here.