Classwork 1 – Vectors I

1. Discuss the notion of a right-handed Cartesian coordinate system in 2D and 3D.

2. Write the following vectors (in 3D space) in component form:

   (a) $2\mathbf{i}+3\mathbf{j}$ (b) $17\mathbf{i}-4\mathbf{j}-\mathbf{k}$

   (c) $\mathbf{j}$ (d) $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

3. Find the magnitudes (lengths) of the vectors in question 2.

4. If a vector **A** has components $(3,-1,-2)$, write **A** in terms of **i**, **j**, and **k**.
   Repeat for a vector with components (0, 0, 7).

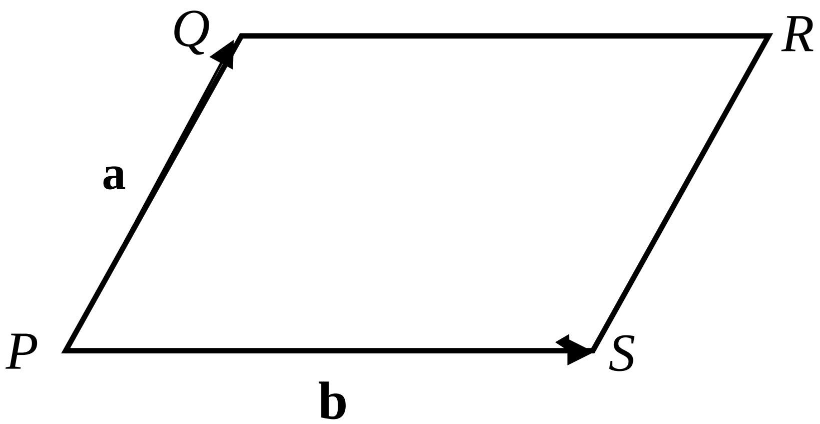
5. If $\mathbf{p}=4\mathbf{i}+2\mathbf{j}$ and $\mathbf{q}=-\mathbf{i}+3\mathbf{k}$, find the vectors
   (a) $-\mathbf{p}$ (b) $2\mathbf{q}$ (c) $\mathbf{p}+\mathbf{q}$ (d) $3\mathbf{p}-5\mathbf{q}$.

6. Find the unit vectors corresponding to the four vectors in (a)-(d) of question 5.

7. The position vectors of points A and B in the *x*-*y* plane have components (2, 1) and (3, 5), respectively. Find (a) the vector $\overrightarrow{AB}$, (b) the vector $\overrightarrow{BA}$, (c) the length of these vectors, and (d) the position vector of the mid-point of $\overrightarrow{AB}$.

8. Point P has position vector $3\mathbf{i} + 5\mathbf{j}$. What is the angle between $\overrightarrow{OP}$ and the *x*-axis?

9. In the parallelogram shown, $\overrightarrow{PQ}=\mathbf{a}$ and $\overrightarrow{PS}=\mathbf{b}$.



   (a) In terms of the vectors **a** and **b**, write down $\overrightarrow{SR}$, $\overrightarrow{RQ}$, $\overrightarrow{PR}$, and $\overrightarrow{SQ}$.

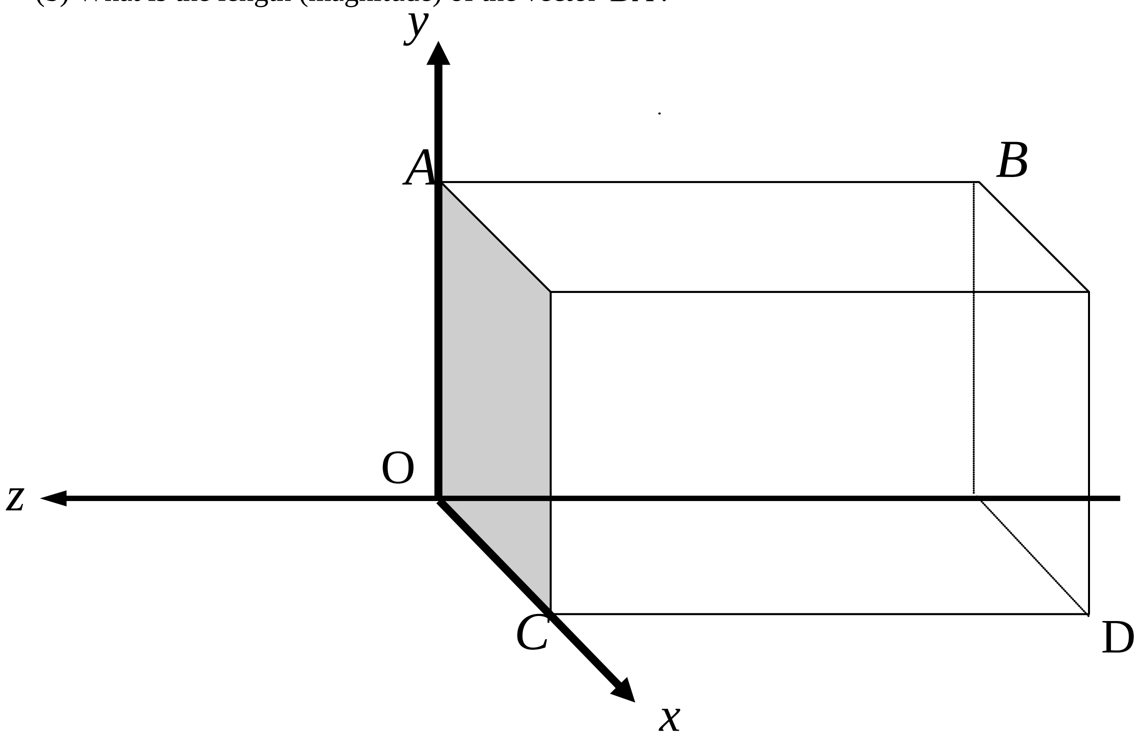
   (b) Defining point *M* as the mid-point of $\overrightarrow{PR}$, and point *N* as the mid-point of $\overrightarrow{SQ}$, find $\overrightarrow{PM}$ and $\overrightarrow{PN}$ in terms of **a** and **b**.

   (c) What property of a parallelogram can you deduce from the result of (b)?

Classwork 2 – Vectors I

1. Let London (L) be at the origin of a 2D Cartesian right-handed coordinate system where the positive direction of the x-axis is east and the positive direction of the *y*-axis is north. The coordinates of Southampton (S) are (–88,–66), and those of Oxford (O) are (–80, 28). Given that the displacement from Bedford (B) to Oxford is $\overrightarrow{BO} = (-52, -42)$ and the displacement from Bedford to Cambridge (C) is $\overrightarrow{BC} = (42, 11)$, find the displacement $\overrightarrow{CS}$ from Cambridge to Southampton. Units are km.

2. In the diagram below, the origin is O, and O, A, B, C and D are five corners of the rectangular box shown. The position vectors of the points B and C are $\mathbf{b} = (0, 8, -14)$ and $\mathbf{c} = (4, 0, 0)$, respectively. (a) Find the vector $\overrightarrow{DA}$ in terms of the natural basis **i**, **j**, and **k**. (b) What is the length (magnitude) of the vector $\overrightarrow{DA}$?



3. Given the three-dimensional vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$, and $\mathbf{c} = -\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, find the vector $\mathbf{d} = \dfrac{3\mathbf{a} - \mathbf{b} + 7\mathbf{c}}{3}$ and its magnitude $|\mathbf{d}|$.

4. Find the angle θ in radians and degrees between the four-dimensional position vectors $\mathbf{a} = (4, 1, -3, 1)$ and $\mathbf{b} = (6, -2, 3, 4)$.

5. Find $\mathbf{a} = (\mathbf{b} \bullet \mathbf{c})\mathbf{d} - \mathbf{e} \times \mathbf{f}$ if $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{e} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{f} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

6. Let $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$. (a) Calculate explicitly $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ (in terms of the coordinates) and show that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. (b) Calculate explicitly $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$. What can you conclude?

Applications in Physics:

7. The force $\mathbf{F}$ on a particle with charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. Explain why the speed $|\mathbf{v}|$ of the particle remains constant.

8. A particle of mass m with position vector $\mathbf{r}$ relative to some origin O experiences a force $\mathbf{F}$, producing a torque $\tau = \mathbf{r}\times\mathbf{F}$ about O. The angular momentum of the particle about O is given by $\mathbf{L} = \mathbf{r}\times m\mathbf{v}$, where $\mathbf{v}$ is the particle's velocity. Show that the rate of change of the angular momentum $\dfrac{d\mathbf{L}}{dt}$ is equal to the applied torque.

Classwork 3

1. Find the equation of the plane that is normal to the vector $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and that passes through the point $\mathbf{a} = (1, -2, 3)$.

2. (a) If the points A = (1, 1, a), B = (2, b, 7) and C = (c, 5, −5) all lie on the plane specified by the equation $4x - 3y + 2z = 7$, find a, b and c.

   (b) Find a unit vector normal to the plane defined in part (a).

   (c) Verify that the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ from part (a) are perpendicular to the normal vector found in part (b).

3. The two planes

$$x + 3y + z = 8$$
$$2x + y + 3z = 7$$

   intersect in a line.

   (a) Find the coordinates of the point where the line intersects the plane $y = 0$.

   (b) Find a vector that is normal to each plane.

   (c) Find a vector directed along the line of intersection.

   (d) Use the results of (a) and (c) to obtain an equation for the line of intersection on vector form and on component form.

4. Evaluate 2×2 the determinant $\begin{vmatrix} a & b \\ ca & cb \end{vmatrix}$ where $c \neq 0$. What can you conclude in general?

5. For each of the following four pairs of equations in two unknowns, write the equations on matrix form and identify those pair of equations that have a unique solution. Use Cramer's rule to solve those that do. For the equations that do not have a unique solution, identify whether they have no solutions or infinitely many solutions.

   (a) $\begin{aligned} 3x + 5y &= 14 \\ 2x + 4y &= 10 \end{aligned}$

   (b) $\begin{aligned} 3x - 5y &= 8 \\ 7x + 2y &= 12 \end{aligned}$

   (c) $\begin{aligned} 6x + 3y &= 9 \\ 4x + 2y &= 6 \end{aligned}$

   (d) $\begin{aligned} 1.4x - 1.2y &= 6.4 \\ -2.1x + 1.8y &= -4.7 \end{aligned}$

6. When the 3×3 determinant $\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is expanded by the first column,

$$\det \mathbf{A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

where the determinant is the sum with alternating signs, of the elements of the first column, each multiplied by the 2×2 determinant obtained by deleting the row and column of the corresponding element. The alternating pattern of signs in front of the elements follows the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

(a) Convince yourself that this result is identical to the one where the determinant is expanded by the first row.

Indeed, one can expand a determinant by any row (column) following the rule that the determinant is the sum with alternating signs, of the elements of the given row (column), each multiplied by the 2×2 determinant obtained by deleting the row and column of the corresponding element.

(b) Expand the determinant by the second row and by the third column and show that they are identical to the result above.

(c) Expand the following 3×3 determinant after the most “convenient” row or column:

$$\begin{vmatrix} 4 & 3 & 7 \\ -5 & 0 & 0 \\ 1.5 & 4 & 2 \end{vmatrix}$$

Classwork 4 – Cramer's Rule & Gauss Elimination

This classwork is about solving a system of linear equations using either Cramer's rule (only applicable for n equations with n unknowns and when $\det\mathbf{A} \neq 0$) or Gauss elimination (always applicable). See page 2 for both methods.

First consider the system of 3 linear equations in three unknowns *x*,*y*,*z*:

$$2x - y + 3z = 9 \qquad \textbf{\textit{(1)}}$$
$$x - y + 4z = 10 \qquad \textbf{\textit{(2)}}$$
$$3x + y + 2z = 6. \qquad \textbf{\textit{(3)}}$$

1. Use Cramer's rule to find x, y, z. Substitute the values of $x, y,$ and z back into the original equations to verify that they are indeed satisfied.

2. Follow these steps to solve the system of linear equations (example of Gauss Elimination).
   (a) Exchange Eqs. (1) and (2).
   (b) Add $(-3)\times$ (the new) Eq.(1) to Eq.(3) to eliminate *x* from that equation.
   (c) Similarly, add $(-2)\times$Eq.(1) to Eq.(2) to eliminate *x* from that equation too.
   (d) Add $(-4)\times$ (the modified) Eq.(2) to Eq.(3) to eliminate *y* from that equation.
   (e) Add Eq.(2) to Eq.(1) and multiply Eq.(3) by 1/10.
   (f) Add $5\times$ Eq. (3) to Eq.(2) and Eq.(3) to Eq,(1).
   (g) Hence, obtain *z* from the final version of Eq.(3), *y* from the final version of Eq.(2), and *x* from the final version of Eq.(1).

3. Solve this system of linear equations using Gauss elimination:

$$x + 2y + z = 7 \qquad \textbf{\textit{(1)}}$$
$$-2x + 3y - z = -5 \qquad \textbf{\textit{(2)}}$$
$$3x + 12y - 6z = 9. \qquad \textbf{\textit{(3)}}$$

   Once again, apply a procedure to eliminate *x* from Eq.(2) both *x* and *y* from Eq.(3).

4. Solve this 4×4 system using Gauss elimination:

$$x_1 + 2x_2 + x_3 + 3x_4 = 18$$
$$2x_1 + 4x_2 + 6x_3 + x_4 = -3$$
$$x_1 + 3x_2 + 5x_4 = 24$$
$$3x_1 + 5x_2 + 2x_3 + 4x_4 = 40.$$

5. There's an obvious "problem" with this system of linear equations. Can you see what it is?

$$2x - y + 3z = 9$$
$$x - y + 4z = 10$$
$$6x - 3y + 9z = 27.$$

Solve the system of linear equations.

6. Solve the system of linear equations:

$$x + 3y - z = 6$$
$$8x + 9y + 4z = 21$$
$$2x + y + 2z = 3.$$

Cramer's Rule (1750, Gabriel Cramer, Swiss mathematician, 1704-1752)

Consider a system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ written on the matrix form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the $n \times n$ matrix of coefficients of the system. If the determinant of $\mathbf{A}$ is different from zero, $\det \mathbf{A} \neq 0$, then there is a unique solution and it is given by

$$x_j = \frac{\det \mathbf{B}^{(j)}}{\det \mathbf{A}} \text{ for } j = 1, 2, \ldots, n,$$

where the matrix $\mathbf{B}^{(j)}$ is the matrix obtained from $\mathbf{A}$ by replacing its jth column with column vector $\mathbf{b}$ making up the right-hand side of the system of equations.

Notice: Cramer's rule is only applicable for a system of n equations with n unknowns. Moreover, Cramer's rule is only applicable when $\det \mathbf{A} \neq 0$, that is, when a unique solution exists. If $\det \mathbf{A} = 0$, no unique solution exists and Cramer's rule does not yield any more information about the system of linear equations. It may have no solutions at all (the set of solutions is empty) or infinitely many solutions (a line, a plane or a hyperplane).

Gauss elimination (1809, Carl Friedrich Gauss, German mathematician,1777-1855)

Gauss elimination is an efficient algorithm for solving a system of linear equations. Gauss elimination is applicable to any system of linear equations, that is, *m* equations with *n* unknown. It will reveal whether there is no solutions, a unique solution, or infinitely many solutions (a line, a plane or, in higher dimensions than 3, a hyperplane).

None of the following operations changes the solution of a system of linear equations:

(i) Changing the order of the equations.

(ii) Multiplying all terms in an equation by the same <u>non-zero</u> constant.

(iii) Adding a multiple $r \in \mathbb{R}$ of any equation to any other equation. (The multiple can be negative, so addition includes subtraction.)

The strategy is to leave only x_n in the last equation, only x_{n-1} and x_n in the next last, and so on.

Classwork 5 – Cramer's Rule & Gauss Elimination

This classwork is about solving a system of linear equations using either Cramer's rule (only applicable for n equations with n unknowns) or Gauss elimination (always applicable). See page 2 for both methods.

First consider the system of 3 linear equations with three unknowns

$$2x - y + 3z = 9 \qquad \textbf{(1)}$$
$$x - y + 4z = 10 \qquad \textbf{(2)}$$
$$3x + y + 2z = 6. \qquad \textbf{(3)}$$

1. Use Cramer's rule to find $x, y, z.$ Substitute the values of $x, y,$ and z back into the original equations to verify that they are indeed satisfied.

2. Follow these steps to solve the system of linear equations (example of Gauss Elimination).
   (a) Exchange Eqs. (1) and (2).
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   (d) Add $(-4)\times$ (the modified) Eq.(2) to Eq.(3) to eliminate *y* from the equation.
   (e) Hence, obtain *z* from the final version of Eq.(3), *y* from the final version of Eq.(2), and *x* from the final version of Eq.(1).

3. Solve this system of linear equations using Gauss elimination:

$$x + 2y + z = 7 \qquad \textbf{(1)}$$
$$-2x + 3y - z = -5 \qquad \textbf{(2)}$$
$$3x + 12y - 6z = 9. \qquad \textbf{(3)}$$

   Once again, apply a procedure to eliminate *x* from Eq.(2) both *x* and *y* from Eq.(3).

4. Solve this 4×4 system using Gauss elimination:

$$x_1 + 2x_2 + x_3 + 3x_4 = 18$$
$$2x_1 + 4x_2 + 6x_3 + x_4 = -3$$
$$x_1 + 3x_2 + 5x_4 = 24$$
$$3x_1 + 5x_2 + 2x_3 + 4x_4 = 40.$$

5. There's an obvious "problem" with this system of linear equations. Can you see what it is?

$$2x - y + 3z = 9$$
$$x - y + 4z = 10$$
$$6x - 3y + 9z = 27.$$

   Solve the system of linear equations.

6. Solve the system of linear equations:

$$\begin{aligned} x+3y-z&=6\\ 8x+9y+4z&=21\\ 2x+y+2z&=3. \end{aligned}$$

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Gauss elimination is an efficient algorithm for solving a system of linear equations. Gauss elimination is applicable to any system of linear equations, that is, *m* equations with *n* unknown. It will reveal whether there is no solutions, a unique solution, or infinitely many solutions (a line, a plane or, in higher dimensions than 3, a hyperplane).

None of the following operations changes the solution of a system of linear equations:

(i) Changing the order of the equations.

(ii) Multiplying all terms in an equation by the same <u>non-zero</u> constant.

(iii) Adding a multiple $r \in \mathbb{R}$ of any equation to any other equation. (The multiple can be negative, so addition includes subtraction.)

The strategy is to leave only x_n in the last equation, only x_{n-1} and x_n in the next last, and so on.

Classwork 6 – Transforming Areas and Volumes

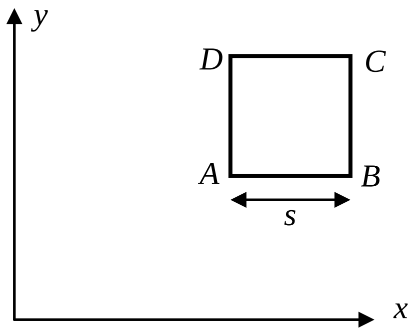
The aim with the following classwork is to illustrate the three general theorems stated on page 2. First we consider a transformation of areas, that is, a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) The matrix $\mathbf{T} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ transforms a point defined by the position vector $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ into a new point $\mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ defined by $\mathbf{r}_2 = \mathbf{T}\mathbf{r}_1$. Write down equations for x_2 and y_2 in terms of x_1 and y_1, and the elements of **T**. How is the origin transformed?

(b) *ABCD* is a square of side *s* as shown in the Fig. with the lower left-hand corner *A* at position $\mathbf{r}_A = \begin{pmatrix} u \\ v \end{pmatrix}$.

Write down expressions for the vectors $\mathbf{r}_B, \mathbf{r}_C$, and $\mathbf{r}_D$ defining the other three corners *B, C,* and *D.*

Find the vectors $\overrightarrow{AB}, \overrightarrow{DC}, \overrightarrow{AD}, \overrightarrow{BC}$.



(c) The vector equation of a straight line through a point $\mathbf{r}_0$ and direction $\mathbf{d}$ is $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}$. Convince yourself that **T** transforms one straight line into another straight line.

(d) It follows from (c) that **T** transforms *ABCD* into a quadrilateral *EFGH*. Write down the position vectors $\mathbf{r}_E, \mathbf{r}_F, \mathbf{r}_G, \mathbf{r}_H$ defining all four corners of *EFGH*, and hence find the vectors $\overrightarrow{EF}, \overrightarrow{HG}, \overrightarrow{EH}, \overrightarrow{FG}$. You should find that $\overrightarrow{EF} = \overrightarrow{HG}$ and $\overrightarrow{EH} = \overrightarrow{FG}$, which implies that opposite sides of the quadrilateral are equal in length and direction, and hence that the quadrilateral is, in fact, a parallelogram.

(e) If $\mathbf{T} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$, $\mathbf{r}_{\mathrm{A}} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, and *s* = 3, find the corners of the parallelogram and the vectors $\overrightarrow{EF}, \overrightarrow{HG}, \overrightarrow{EH}, \overrightarrow{FG}$. Make a rough sketch of the square *ABCD* and its transform *EFGH*.

(f) The area of a parallelogram is $|\mathbf{A} \times \mathbf{B}|$ where **A** and **B** are the two vectors defining the two adjacent sides. The parallelogram lies in the *x-y* plane, that is, $A_z = B_z = 0$, and hence the area is $|A_x B_y - A_y B_x|$.

(i) Find the area of *EFGH*, and show that the area scale factor for the transformation **T**, i.e, the factor by which the area of the original square is multiplied, is $|\det \mathbf{T}| = |a_1 b_2 - a_2 b_1|$.

(ii) Put in the numbers from the previous question.

(g) Does the same area scale factor apply to other 2D shapes transformed by **T**?

Now we consider a transformation of volumes, that is, a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(h) These ideas above can be extended to 3D (indeed to any dimension). A unit cube has edges of unit length parallel to the coordinate axes and one corner is at the origin. The linear transformation $\mathbf{T} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ transforms the cube into a parallelepiped.

(i) Write down the vectors representing the three edges of the parallelepiped that intersect at the origin.

(ii) By using the formula from the Lecture notes for the volume of a parallelepiped, find the volume scale factor for the transformation applied to the cube.

(iii) Does the same volume scale factor apply to other three-dimensional shapes transformed by **T**?

The examples above in 2D and 3D illustrate the following theorems valid for any dimension n – as you can see, the determinant is indeed a very valuable concept:

Theorem 1: A linear function f from $\mathbb{R}^n$ to $\mathbb{R}^n$ with matrix $\mathbf{A}$ multiplies volumes by the factor $|\det \mathbf{A}|$.

Theorem 2: Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then the associated matrix

$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \equiv (\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n)$ where the jth column $\mathbf{a}_j = f(\mathbf{e}_j),\ j = 1, 2, \ldots n$ is the transformation of the jth natural basis vector $\mathbf{e}_j$.

Theorem 3: Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n vectors in $\mathbb{R}^n$. Then the volume of the parallelepiped with edges $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is $|\det(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)|$.

Classwork 7 – Discover the Orthogonal Matrix

Definition 1: A unit vector $\hat{\mathbf{a}} \in \mathbb{R}^n, |\hat{\mathbf{a}}| = 1$ is called a *normalised* vector.

Definition 2: Two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ that are perpendicular to each other, that is, $\mathbf{a} \bullet \mathbf{b} = 0$, are called *orthogonal*.

Definiton 3: Two unit vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathbb{R}^n$ that are perpendicular to each other, $\hat{\mathbf{a}} \bullet \hat{\mathbf{b}} = 0$, are called *orthonormal*.

This present classwork, leads you to the definition of an *orthogonal matrix*. Although the questions relate to two-dimensional vectors and 2×2 matrices, the results are valid in $\mathbb{R}^n$.

(a) (i) Show that the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ represented by the 2×1 matrices $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ are orthonormal if $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1$ and $u_x v_x + u_y v_y = 0$.

(ii) Show that these conditions can be expressed in the matrix form $\mathbf{u}^t\mathbf{u} = \mathbf{v}^t\mathbf{v} = 1$ and $\mathbf{u}^t\mathbf{v} = \mathbf{v}^t\mathbf{u} = 0$ where $\mathbf{A}^t$ denotes the transpose of the matrix $\mathbf{A}$.

(b) Find the unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, and find two other unit vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ that are orthonormal to $\hat{\mathbf{a}}$.

(c) Consider the 2×2 matrix $\mathbf{O}$ made up of the two orthonormal vectors $\mathbf{u}, \mathbf{v}$ from part (a), that is, $\mathbf{O} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$. Show that $\mathbf{O}^t\mathbf{O} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 4: A matrix $\mathbf{O}$ satisfying $\mathbf{O}^t\mathbf{O} = \mathbf{I}$ is called an *orthogonal matrix*.

(d) We will now discover some additional properties of orthogonal matrices.

(i) Consider two vectors $\mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$ with $\mathbf{q} = \mathbf{A}\mathbf{p}$ where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. If the magnitudes of $\mathbf{p}$ and $\mathbf{q}$ are identical, what conditions are imposed on the elements of $\mathbf{A}$?

(ii) Which of the following statements is correct: (1) $\mathbf{q}^t = \mathbf{A}^t\mathbf{p}^t$ or (2) $\mathbf{q}^t = \mathbf{p}^t\mathbf{A}^t$?

(e) Consider two vectors $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^2$ which are transformed (mapped) by the matrix $\mathbf{A}$ into $\mathbf{q}_1, \mathbf{q}_2$, that is, $\mathbf{q}_1 = \mathbf{A}\mathbf{p}_1$ and $\mathbf{q}_2 = \mathbf{A}\mathbf{p}_2$, respectively. If the transformation $\mathbf{A}$ does not change the scalar product, that is, $\mathbf{p}_1 \bullet \mathbf{p}_2 = \mathbf{q}_1 \bullet \mathbf{q}_2$ or, in matrix form, $\mathbf{p}_1^t \mathbf{p}_2 = \mathbf{q}_1^t \mathbf{q}_2$, show that $\mathbf{A}^t \mathbf{A} = \mathbf{I}$ in other words that $\mathbf{A}$ is an orthogonal matrix.
The situation in part (d) is the special case where $\mathbf{p}_2 = \mathbf{p}_1$ and $\mathbf{q}_2 = \mathbf{q}_1$.

(f) Is the rotation matrix $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ an orthogonal matrix? Qualify your answer.

(g) Let $\hat{\mathbf{a}}$ and each of the vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ from part (b) in turn form two orthogonal matrices $\mathbf{O}_1$ and $\mathbf{O}_2$ like $\mathbf{O}$ in part (c). If the transformation represents a rotation, find the angle. If not, try to figure out what the operation does represent.

(h) Transform the vector $\mathbf{s} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ using each orthogonal matrix from part (g) in turn. Check that the new vectors $\mathbf{t}_1 = \mathbf{O}_1\mathbf{s}$ and $\mathbf{t}_2 = \mathbf{O}_2\mathbf{s}$ have the same magnitude as **s**. Find the angle between **s** and each of the vectors $\mathbf{t}_1, \mathbf{t}_2$, and draw all three vectors on a diagram.

Classwork 8: Eigenvalue Problem

1. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

(i) Find the eigenvalues λ_1, λ_2 with $\lambda_1 > \lambda_2$.

(ii) How are the eigenvalues related to the determinant $\det \mathbf{A}$ and the trace $Trace\mathbf{A}$ of the matrix **A**?

(iii) Find the associated eigevectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of the matrix **A**.

(iv) Show that the eigenvectors are orthogonal (perpendicular) and comment on the result.

(v) Construct the matrix of eigenvectors **S** and show explicitly that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \Lambda = diag(\lambda_1, \lambda_2)$.

2. Find the eigenvalues and eigenvectors of (i) $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$, (ii) $\mathbf{B} = \begin{pmatrix} 5 & -7 \\ 1 & -3 \end{pmatrix}$.

In one case, the eigenvectors are *orthogonal*. Identify this case and explain how this fact related to the structure of the matrix.

Considering this case only,

(iii) Make the set of eigenvectors *orthonormal.*

(iv) Create an orthogonal matrix and show explicitly that $\mathbf{S}^{-1} = \mathbf{S}^t$.

(v) Calculate $\mathbf{A}^{247}$.

3. With $\mathbf{B} = \begin{pmatrix} 5 & -7 \\ 1 & -3 \end{pmatrix}$, deduce the eigenvalues of $\mathbf{B}^2$.

4. Find the eigenvalues and eigenvectors of the transformations

(i) $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 27 \end{pmatrix}$, (ii) $\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, (iii) $\mathbf{C} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}$, (iv) $\mathbf{D} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Un-normalised eigenvectors will do. Note that generally for 3×3 matrices, the characteristic equation will yield a cubic equation for the eigenvalues. Once you have found one of them, λ_1 (maybe by inspection) you can divide the cubic equation through by $(\lambda - \lambda_1)$ and solve the resulting quadratic equation to find λ_2 and λ_3.

Classwork 1: Answers

1. See Section 1 in Lecture notes for a discussion. Make sure you know the rules to determine whether a Cartesian coordinate system is right-handed or left-handed.

2. In general (in 3D), $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z)$ so we find
   (a) (2, 3, 0) (b) (17, −4, −1)
   (c) (0, 1, 0) (d) (x, y, z)

3. In general (in 3D), the magnitude of a vector $\mathbf{r} = (x, y, z)$ is $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$:
   (a) $\sqrt{2^2+3^2+0^2} = \sqrt{13}$ (b) $\sqrt{17^2+(-4)^2+(-1)^2} = \sqrt{306}$
   (c) $\sqrt{0^2+1^2+0^2} = 1$ (d) $\sqrt{x^2+y^2+z^2}$

4. $\mathbf{A} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 7\mathbf{k}$.

5. (a) $-\mathbf{p} = -4\mathbf{i} - 2\mathbf{j}$ (b) $2\mathbf{q} = -2\mathbf{i} + 6\mathbf{k}$ (c) $\mathbf{p}+\mathbf{q} = (4\mathbf{i}+2\mathbf{j}) + (-\mathbf{i}+3\mathbf{k}) = 3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$
   (d) $3\mathbf{p} - 5\mathbf{q} = 3(4\mathbf{i}+2\mathbf{j}) - 5(-\mathbf{i}+3\mathbf{k}) = 17\mathbf{i}+6\mathbf{j}-15\mathbf{k}$

6. (a) $|-\mathbf{p}| = |\mathbf{p}| = \sqrt{4^2+2^2} = 2\sqrt{5} \Rightarrow \dfrac{-\mathbf{p}}{|-\mathbf{p}|} = \dfrac{-2\mathbf{i}-\mathbf{j}}{\sqrt{5}}$
   (b) $|2\mathbf{q}| = \sqrt{(-2)^2+6^2} = 2\sqrt{10} \Rightarrow \dfrac{2\mathbf{q}}{|2\mathbf{q}|} = \dfrac{-\mathbf{i}+3\mathbf{k}}{\sqrt{10}}$
   (c) $|\mathbf{p}+\mathbf{q}| = \sqrt{3^2+2^2+3^2} = \sqrt{22} \Rightarrow \dfrac{\mathbf{p}+\mathbf{q}}{|\mathbf{p}+\mathbf{q}|} = \dfrac{3\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{22}}$
   (d) $|3\mathbf{p}-5\mathbf{q}| = \sqrt{17^2+6^2+(-15)^2} = \sqrt{550} \Rightarrow \dfrac{3\mathbf{p}-5\mathbf{q}}{|3\mathbf{p}-5\mathbf{q}|} = \dfrac{17\mathbf{i}+6\mathbf{j}-15\mathbf{k}}{\sqrt{550}}$

7. (a) $(3,5)-(2,1) = (1,4)$ ***or*** $\mathbf{i}+4\mathbf{j}$ (b) $(2,1)-(3,5) = (-1-4)$ or $-\mathbf{i}-4\mathbf{j}$
   (c) $|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{1^2+4^2} = \sqrt{17}$ (d) $\overrightarrow{OA} + \dfrac{1}{2}\overrightarrow{AB} = (2\mathbf{i}+\mathbf{j}) + \dfrac{1}{2}(\mathbf{i}+4\mathbf{j}) = 2.5\mathbf{i}+3\mathbf{j}$

8. Call the angle θ. Then $\tan\theta = 5/3 \Rightarrow \theta = \tan^{-1}(5/3) = 1.030\,\text{rad} = 59.04^\circ$.

9. (a) $\overrightarrow{SR} = \mathbf{a}, \overrightarrow{RQ} = -\mathbf{b}, \overrightarrow{PR} = \mathbf{a}+\mathbf{b}, \overrightarrow{SQ} = \mathbf{a}-\mathbf{b}.$

   (b) $\overrightarrow{PM} = \dfrac{1}{2}\overrightarrow{PR} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b})$ $\overrightarrow{PN} = \mathbf{b} + \dfrac{1}{2}\overrightarrow{SQ} = \mathbf{b} + \tfrac{1}{2}(\mathbf{a}-\mathbf{b}) = \tfrac{1}{2}(\mathbf{a}+\mathbf{b})$

   (c) The diagonals of a parallelogram bisect one another.

Classwork 2: Answers

1. Clearly, $\overrightarrow{CS} = \overrightarrow{LS} - \overrightarrow{LC}$ where $\overrightarrow{LS}$ and $\overrightarrow{LC}$ denote the position vectors of Southampton and Cambridge, respectively. We must determine $\overrightarrow{LC}$ and we find $\overrightarrow{LC} = \overrightarrow{LO} + \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{LO} - \overrightarrow{BO} + \overrightarrow{BC} = (-80+52+42, 28+42+11) = (14,81)$ such that $\overrightarrow{CS} = \overrightarrow{LS} - \overrightarrow{LC} = (-88,-66)-(14,81) = (-102,-147)$.

2. (a) We identify the position vectors for points A and D as $\mathbf{a} = (0,8,0)$ and $\mathbf{d} = (4,0,-14)$, respectively. Hence $\overrightarrow{DA} = \mathbf{a} - \mathbf{d} = (-4,8,14) = -4\mathbf{i} + 8\mathbf{j} + 14\mathbf{k}$.

   (b)The length $\left|\overrightarrow{DA}\right| = \sqrt{(-4)^2 + 8^2 + 14^2} = \sqrt{276} \approx 16.6$.

3. (a) We find $3\mathbf{a} - \mathbf{b} + 7\mathbf{c} = \left(12-5-7\right)\mathbf{i} + (-9-2-7)\mathbf{j} + (9+6-21)\mathbf{k} = -18\mathbf{j} - 6\mathbf{k}$ that is, $\mathbf{d} = \frac{1}{3}(3\mathbf{a} - \mathbf{b} + 7\mathbf{c}) = -6\mathbf{j} - 2\mathbf{k}$. (b) $|\mathbf{d}| = \sqrt{(-6)^2 + (-2)^2} = \sqrt{40} \approx 6.32$.

4. The magnitudes of the two vectors are $|\mathbf{a}| = \sqrt{4^2 + 1^2 + (-3)^2 + 1^2} = \sqrt{27}$ and $|\mathbf{b}| = \sqrt{6^2 + (-2)^2 + 3^2 + 4^2} = \sqrt{65}$, respectively and their dot-product (scalar product) $\mathbf{a}\bullet\mathbf{b} = 4\cdot 6 + 1\cdot(-2) + (-3)\cdot 3 + 1\cdot 4 = 17$ from which we find $\cos\theta = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{17}{\sqrt{27}\sqrt{65}}$,implying that the angle $\theta \approx 1.153 \text{ rad} \approx 66.06^{o}$ $(0 \le \theta \le \pi)$.

5. The scalar product $\mathbf{b}\bullet\mathbf{c} = 2\cdot 4 + (-3)\cdot(-3) + 3\cdot(-4) = 5.$ Using the method of the determinant to compute the vector (cross) product, we find that:

$$\mathbf{e}\times\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ -1 & 2 & 2 \end{vmatrix} = ((-3)\cdot 2 - 2\cdot 2)\mathbf{i} - (1\cdot 2 - (-1)\cdot 2)\mathbf{j} + (1\cdot 2 - (-1)\cdot(-3))\mathbf{k} = -10\mathbf{i} - 4\mathbf{j} - \mathbf{k}.$$

   . Hence we find that $\mathbf{a} = (\mathbf{b}\bullet\mathbf{c})\mathbf{d} - \mathbf{e}\times\mathbf{f} = 5(-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - (-10\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 19\mathbf{j} - 4\mathbf{k}.$

6. (a) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$.

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ a_x & a_y & a_z \end{vmatrix} = (a_z b_y - a_y b_z)\mathbf{i} - (a_z b_x - a_x b_z)\mathbf{j} + (a_y b_x - a_x b_y)\mathbf{k}$$. Hence $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

   Due to the minus sign, we say that the vector (cross) product is anti-commutative.

   (b) $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = a_x(a_y b_z - a_z b_y) - a_y(a_x b_z - a_z b_x) + a_z(a_x b_y - a_y b_x) = 0$ and similarly

   $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = b_x(a_y b_z - a_z b_y) - b_y(a_x b_z - a_z b_x) + b_z(a_x b_y - a_y b_x) = 0$. Since the scalar products are zero, we can conclude that $\mathbf{a} \perp (\mathbf{a}\times\mathbf{b})$ and $\mathbf{b} \perp (\mathbf{a}\times\mathbf{b})$.

Applications in physics:

7. Since $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, the force $\mathbf{F} \perp \mathbf{v}$ (and also, of course $\mathbf{F} \perp \mathbf{B}$) and hence there is no work done on the particle. There is no change in the kinetic energy and therefore the speed $|\mathbf{v}|$ remains constant.

8. The torque $\tau = \mathbf{r} \times \mathbf{F}$ about O and the angular momentum of the particle about O is given by $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$. Using the rule of differentiating a vector product, we find that the rate of change of the angular momentum $\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \left(\frac{d\mathbf{r}}{dt}\right) \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \mathbf{F}$ since $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{F} = m\mathbf{a}$.

Classwork 3: Answers

1. Let $\mathbf{r}$ be any point in the plane passing through $\mathbf{a}$. Then the vector $\mathbf{r}-\mathbf{a}$ is in the plane and hence normal to $\mathbf{n}$, implying $(\mathbf{r}-\mathbf{a})\bullet\mathbf{n}=0$ or $\mathbf{r}\bullet\mathbf{n}=\mathbf{a}\bullet\mathbf{n}$. We note that $\mathbf{r}\bullet\mathbf{n}=x+y-2z$ while $\mathbf{a}\bullet\mathbf{n}=1\cdot1+(-2)\cdot1+3\cdot(-2)=-7$ The equation of the plane is therefore $x+y-2z=-7$, or any multiple thereof.

2. (a) Since the points A, B, and C are in the plane, they must satisfy the equation $4x-3y+2z=7$. Substituting the coordinates of A, B and C in turn into the equation yields $4-3+2a=7 \Leftrightarrow a=3$, $8-3b+14=7 \Leftrightarrow b=5$, and $4c-15-10=7 \Leftrightarrow c=8$.

   (b) From the equation of the plane $4x-3y+2z=7$ we identify that the vector $\mathbf{n}=4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ is normal to the plane. Its magnitude is $|\mathbf{n}|=\sqrt{4^2+(-3)^2+2^2}=\sqrt{29}$ so the unit normal vector is $\pm\hat{\mathbf{n}}=\pm\frac{\mathbf{n}}{|\mathbf{n}|}=\pm\frac{4\mathbf{i}-3\mathbf{j}+2\mathbf{k}}{\sqrt{29}}$. Either sign is acceptable.

   (c) $\overrightarrow{AC}=(8,5,-5)-(1,1,3)=7\mathbf{i}+4\mathbf{j}-8\mathbf{k}$ implying $\overrightarrow{AC}\bullet\mathbf{n}=7\cdot4+4\cdot(-3)+(-8)\cdot2=0$. Also, $\overrightarrow{BC}=(8,5,-5)-(2,5,7)=6\mathbf{i}-12\mathbf{k}$ implying $\overrightarrow{BC}\bullet\mathbf{n}=6\cdot4+0\cdot(-3)+(-12)\cdot2=0$. Hence we conclude that $\overrightarrow{AC}\perp\mathbf{n}$ and $\overrightarrow{BC}\perp\mathbf{n}$.

3. (a) Substituting $y=0$ into the two equations yields $x+z=8$ and $2x+3z=7$, with the solution $x=17, z=-9$. Therefore, the coordinates of the point where the line intersects the plane $y=0$ is $(17,0,-9)$ or $\mathbf{r_0}=17\mathbf{i}-9\mathbf{k}$.

   (b) The coefficients in front of x,y,z determine a normal vector to each plane. Hence $\mathbf{n}_1=\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\mathbf{n}_2=2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ or, indeed, multiples thereof are normal vectors to plane 1 and plane two, respectively.

   (c) A vector $\mathbf{d}$ directed along the line of intersection clearly lies in both planes and therefore it is perpendicular to both the normal vectors. Therefore, we may use $\mathbf{d}=\mathbf{n}_1\times\mathbf{n}_2$.

   Taking the cross product we find $\mathbf{d}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & 1\\ 2 & 1 & 3\end{vmatrix}=8\mathbf{i}-\mathbf{j}-5\mathbf{k}$.

   (d) The equation for the line of intersection is determined by a point on the line, $\mathbf{r}_0$, and a direction vector for the line, $\mathbf{d}$. Hence, the equation for the line of intersection is $\mathbf{r}=\mathbf{r}_0+\lambda\mathbf{d}=17\mathbf{i}-9\mathbf{k}+\lambda(8\mathbf{i}-\mathbf{j}-5\mathbf{k})=(17+8\lambda)\mathbf{i}-\lambda\mathbf{j}-(9+5\lambda)\mathbf{k}$ or, on component form: $x=17+8\lambda; y=0-\lambda; z=-9-5\lambda$. Isolating λ from the three associated equations, we find $\lambda=\frac{x-17}{8}=\frac{y}{-1}=\frac{z+9}{-5}$. Note the direction ratios given by the coordinates of $\mathbf{d}$ from part (c) are in the denominators, and the coordinates of $\mathbf{r}_0$ from part (a) in the numerators.

4. $\begin{vmatrix}a & b\\ ca & cb\end{vmatrix}=a\cdot cb-ca\cdot b=0$. Indeed, if any two rows (or colums) are proportional in a determinant, it is 0.

5. (a) $\begin{matrix} 3x+5y=14 \\ 2x+4y=10 \end{matrix} \Leftrightarrow \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$. The determinant of the matrix of coefficients is $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 3\cdot 4 - 2\cdot 5 = 2 \neq 0$. Hence there is a unique solution and according to Cramer's rule $x = \dfrac{\begin{vmatrix} 14 & 5 \\ 10 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \dfrac{14\cdot 4 - 10\cdot 5}{2} = \dfrac{6}{2} = 3$ and $y = \dfrac{\begin{vmatrix} 3 & 14 \\ 2 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \dfrac{3\cdot 10 - 2\cdot 14}{2} = \dfrac{2}{2} = 1$. The two lines cross at the point $(x, y) = (3,1)$. Test this graphically ☺.

(b) $\begin{matrix} 3x-5y=8 \\ 7x+2y=12 \end{matrix} \Leftrightarrow \begin{pmatrix} 3 & -5 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$. The determinant of the matrix of the coefficients to the system of linear equations is $\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix} = 3\cdot 2 - 7\cdot(-5) = 41 \neq 0$. There is a unique solution. Cramer's rule yields $x = \dfrac{\begin{vmatrix} 8 & -5 \\ 12 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix}} = \dfrac{8\cdot 2 - 12\cdot(-5)}{41} = \dfrac{76}{41}$ and $y = \dfrac{\begin{vmatrix} 3 & 8 \\ 7 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix}} = \dfrac{3\cdot 12 - 7\cdot 8}{41} = -\dfrac{20}{41}$. The two lines cross at the point $(x, y) = \left(\dfrac{76}{41}, -\dfrac{20}{41}\right)$.

(c) $\begin{matrix} 6x+3y=9 \\ 4x+2y=6 \end{matrix} \Leftrightarrow \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$ The determinant of the matrix of coefficients is $\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 6\cdot 2 - 4\cdot 3 = 0$. Hence there is no unique solution. We notice that the two equations are proportional since the second can be obtained from the first by multiplication with 2/3. Therefore, the equations represent the same line and we have infinitely many solutions, namely all the points on the line $2x + y = 3.$.

(d) The associated determinant $\begin{vmatrix} 1.4 & -1.2 \\ -2.1 & 1.8 \end{vmatrix} = 1.4\cdot 1.8 - (-2.1)\cdot(-1.2) = 0$. Hence there is no unique solution. The two lines are parallel but have no points in common and there are no solutions to the pair of equations.

6. (a) Expanding the determinant of $\mathbf{A}$ by the first row yields

$$\begin{aligned}\det\mathbf{A} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}).\end{aligned}$$

where we have re-arranged the last two terms. Inspection shows that this is indeed equal to the determinant expanded by the first column.

(b) Let us expand the determinant of $\mathbf{A}$ by the second row. Using the alternating pattern for the sign, we find

$$\begin{aligned}\det\mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} \\ &= -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix} \\ &= -a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{23}(a_{11}a_{32} - a_{31}a_{12}) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}).\end{aligned}$$

Let us expand the determinant of $\mathbf{A}$ by the third column. Again, using the sign-pattern:

$$\begin{aligned}\det\mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} \\ &= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} \\ &= a_{13}(a_{21}a_{32} - a_{31}a_{22}) - a_{23}(a_{11}a_{32} - a_{31}a_{12}) + a_{33}(a_{11}a_{22} - a_{21}a_{12}) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}).\end{aligned}$$

Indeed, the determinant of $\mathbf{A}$ may be expanded by any row or any column. All the results are identical. Hence, one would often choose to evaluate a determinant by expanding by the row or column that has a maximum number of zeros to minimise the algebra. ☺

(c) For example, by expanding the following determinant by the second row, we find

$$\begin{vmatrix} 4 & 3 & 7 \\ -5 & 0 & 0 \\ 1.5 & 4 & 2\end{vmatrix} = -(-5)\begin{vmatrix} 3 & 7 \\ 4 & 2\end{vmatrix} = 5\cdot(6-28) = -110$$. Note the sign!

Mathematics: Linear Algebra - Classwork 4 - Answers, 2/11/2010

1. The matrix of coefficients

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 4 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{so} \quad \det\mathbf{A} = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -1 & 4 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 5 \\ 4 & 0 & 6 \\ 3 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 5 & 5 \\ 4 & 6 \end{vmatrix} = -10.$$

R3+R1, R3+R2 Expanding by C2

Because $\det\mathbf{A} \neq 0$, a unique solution exists. We apply Cramer's rule to find x, y, z.

$$\det\mathbf{B}^{(1)} = \begin{vmatrix} 9 & -1 & 3 \\ 10 & -1 & 4 \\ 6 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 15 & 0 & 5 \\ 16 & 0 & 6 \\ 6 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 15 & 5 \\ 16 & 6 \end{vmatrix} = -10$$

R3+R1, R3+R2 Expand by C2

$$\det\mathbf{B}^{(2)} = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 10 & 4 \\ 3 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -11 & -5 \\ 1 & 10 & 4 \\ 0 & -24 & -10 \end{vmatrix} = -1 \begin{vmatrix} -11 & -5 \\ -24 & -10 \end{vmatrix} = 10$$

-2R2+R1, -3R2+R3 Expand by C1

$$\det\mathbf{B}^{(3)} = \begin{vmatrix} 2 & -1 & 9 \\ 1 & -1 & 10 \\ 3 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 15 \\ 4 & 0 & 16 \\ 6 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 5 & 15 \\ 4 & 16 \end{vmatrix} = -20$$

R3+R1, R3+R2 Expand by C2

Hence, Cramer's rule yields

$$(x, y, z) = \left(\frac{\det\mathbf{B}^{(1)}}{\det\mathbf{A}}, \frac{\det\mathbf{B}^{(2)}}{\det\mathbf{A}}, \frac{\det\mathbf{B}^{(3)}}{\det\mathbf{A}} \right) = (1, -1, 2)$$

Substituting into the original euations, we find that they are satisfied:

$$\begin{aligned} 2 \cdot 1 - (-1) + 3 \cdot 2 &= 9 \\ 1 - (-1) + 4 \cdot 2 &= 10 \\ 3 \cdot 1 + (-1) + 2 \cdot 2 &= 6. \end{aligned}$$

2. We work on the augmented matrix associated with the system of linear equations, that is, the matrix obtained from the matrix of coefficients by adding the RHS as an extra column: Applying Gaussian elimination on the same set of equations, we find:

$$\text{Exchange R1 and R2} \left(\begin{array}{ccc|c} 2 & -1 & 3 & 9 \\ 1 & -1 & 4 & 10 \\ 3 & 1 & 2 & 6 \end{array} \right) \Leftrightarrow \text{-3R1+R3, -2R1+R2} \left(\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 2 & -1 & 3 & 9 \\ 3 & 1 & 2 & 6 \end{array} \right) \Leftrightarrow$$

$$\text{-4R2+R3, R2+R1} \left(\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & 1 & -5 & -11 \\ 0 & 4 & -10 & -24 \end{array} \right) \Leftrightarrow \frac{1}{10}\text{R3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -5 & -11 \\ 0 & 0 & 10 & 20 \end{array} \right) \Leftrightarrow$$

$$\text{5R3+R2, R3+R1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -5 & -11 \\ 0 & 0 & 1 & 2 \end{array} \right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

and we find that there is a unique solution $(x, y, z) = (1, -1, 2)$.

3. We apply Gauss elimination, working on the augmented matrix:

$$\text{2R1+R2, -3R1+R3}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ -2 & 3 & -1 & -5 \\ 3 & 12 & -6 & 9 \end{array}\right) \Leftrightarrow \frac{1}{6}\text{R3, Exchange R2 and R3}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 7 & 1 & 9 \\ 0 & 6 & -9 & -12 \end{array}\right) \Leftrightarrow$$

$$\text{-7R2+R3, -2R2+R1}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -1.5 & -2 \\ 0 & 7 & 1 & 9 \end{array}\right) \Leftrightarrow \frac{1}{11.5}\text{R3}\left(\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & 1 & -1.5 & -2 \\ 0 & 0 & 11.5 & 23 \end{array}\right) \Leftrightarrow$$

$$\text{1.5R3+R2, -4R3+R1}\left(\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & 1 & -1.5 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

and we find that there is a unique solution $(x, y, z) = (3, 1, 2)$. We can easily check that this is indeed a solution by substituting into the original system of equations.

4. We apply Gauss elimination, working on the augmented matrix:

$$\text{-2R1+R2, -R1+R3, -3R1+R4}\left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 18 \\ 2 & 4 & 6 & 1 & -3 \\ 1 & 3 & 0 & 5 & 24 \\ 3 & 5 & 2 & 4 & 40 \end{array}\right) \Leftrightarrow \text{-2R3+R1, R3+R4}\left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 18 \\ 0 & 0 & 4 & -5 & -39 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & -1 & -1 & -5 & -14 \end{array}\right) \Leftrightarrow$$

$$\text{2R4+R2}\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 6 \\ 0 & 0 & 4 & -5 & -39 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & -2 & -3 & -8 \end{array}\right) \Leftrightarrow -\frac{1}{11}\text{R2, Exchange R2, R3, R4}\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 6 \\ 0 & 0 & 0 & -11 & -55 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & -2 & -3 & -8 \end{array}\right) \Leftrightarrow$$

$$\text{R4+R1, -2R4+R2, -3R4+R3}\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 6 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & -2 & -3 & -8 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right) \Leftrightarrow \frac{3}{2}\text{R3+R1}, -\frac{1}{2}\text{R3+R2}, -\frac{1}{2}\text{R3}\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 11 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & -2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right) \Leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{43}{2} \\ 0 & 1 & 0 & 0 & -\frac{15}{2} \\ 0 & 0 & 1 & 0 & -\frac{7}{2} \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

and we find that there is a unique solution $(x_1, x_2, x_3, x_4) = \left(\frac{43}{2}, -\frac{15}{2}, -\frac{7}{2}, 5\right)$. We can easily check that this is indeed a solution by substituting into the original system of equations.

5. The third equation is $3\times$ the first and hence these two equations describe the same plane. Therefore, the system has no unique solution. Indeed, the determinant of the coefficients is zero because the first row is proportional to the thrid row. We conclude that Eq.(3) (or Eq.(1)) is redundant.

   We apply Gauss elimination, working on the augmented matrix:

$$\text{-3R1+R3, Exchange R1 and R2}\left(\begin{array}{ccc|c} 2 & -1 & 3 & 9 \\ 1 & -1 & 4 & 10 \\ 6 & -3 & 9 & 27 \end{array}\right) \Leftrightarrow \text{-2R1+R2}\left(\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 2 & -1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow$$

$$\text{R2+R1}\left(\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & 1 & -5 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -5 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

that is,

$$\begin{cases} x - z &= -1, \\ y - 5z &= -11 \end{cases}$$

as the last equation is redundant. Hence we have 2 equations in 3 unknowns, that is, one degree of freedom. The solution is a line. We assign an arbitrary value $\lambda \in \mathbb{R}$ to z and we find

$$\begin{cases} x &= -1 + \lambda, \\ y &= -11 + 5\lambda, \\ z &= \lambda, \end{cases} \quad \text{or on parametric vector form} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -11 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \mathbf{r}_0 + \lambda \mathbf{d} \quad \text{where } \lambda \in \mathbb{R}.$$

This is a line passing through the point $\mathbf{r}_0$ along the direction $\mathbf{d}$. If you prefer, isolate λ to find

$$\lambda = \frac{z-0}{1} = \frac{y+11}{5} = \frac{x+1}{1}.$$

Admittedly, you were not asked to obtain the result on component form, but who can resits :-)

6. We evaluate the determinant of the matrix of coefficients:

$$\det\mathbf{A} = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 9 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 12 & 21 & 0 \\ 4 & 7 & 0 \end{vmatrix} = -1 \begin{vmatrix} 12 & 21 \\ 4 & 7 \end{vmatrix} = 0.$$

4R1+R2, 2R1+R3 Expanding by C3

Hence, according to Cramer's rule, there is no unique solution.We apply Gauss elimination to reveal whether there are no solutions or infinitely many solutions. Working on the augmented matrix we find

$$\text{-8R1+R2, -2R1+R3} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 8 & 9 & 4 & 21 \\ 2 & 1 & 2 & 3 \end{array}\right) \Leftrightarrow \frac{1}{5}\text{R2 + R1}, -\frac{1}{3}\text{R2+R3} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 0 & -15 & 12 & -27 \\ 0 & -5 & 4 & -9 \end{array}\right) \Leftrightarrow$$

$$-\frac{1}{15}\text{R2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{7}{5} & \frac{3}{5} \\ 0 & -15 & 12 & -27 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{7}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{9}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

that is,

$$\begin{cases} x + \frac{7}{5}z &= \frac{3}{5}, \\ y - \frac{4}{5}z &= \frac{9}{5} \end{cases}$$

as the last equation is redundant. Hence we have 2 equations in 3 unknowns, that is, one degree of freedom. The solution is a line. We assign an arbitrary value $\lambda \in \mathbb{R}$ to z and we find

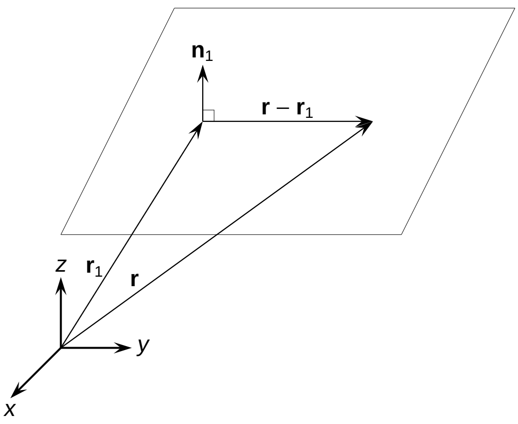
$$\begin{cases} x &= \frac{3}{5} - \frac{7}{5}\lambda, \\ y &= \frac{9}{5} + \frac{4}{5}\lambda, \\ z &= \lambda, \end{cases} \quad \text{or on parametric vector form} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{9}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{7}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} = \mathbf{r}_0 + \lambda \mathbf{d} \quad \text{where } \lambda \in \mathbb{R}.$$

This is a line passing through the point $\mathbf{r}_0$ along the direction $\mathbf{d}$.

Mathematics: Linear Algebra - Classwork 5 - Answers, 5/11/2010

1. (i) The vector $\mathbf{r} - \mathbf{r}_1$ lies in the plane. Therefore, it is perpendicular to the normal vector $\mathbf{n}_1$ of the plane and hence

$$(\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0.$$



[2 marks]

(ii) By inserting the values for $\mathbf{r}_1$ and $\mathbf{n}_1$, we find that

$$\begin{pmatrix} x-3 \\ y-1 \\ z-2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = (x-3)\cdot 2 + (y-1)\cdot(-1) + (z-2)\cdot 4 = 0 \Leftrightarrow 2x - y + 4z = 13.$$

[1 mark]

(iii) We can write the system of linear equations on the matrix form

$$\begin{pmatrix} 2 & -1 & 4 \\ \alpha & 1 & -1 \\ -1 & 1 & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ 5 \end{pmatrix} \Leftrightarrow \mathbf{A}_\alpha \mathbf{r} = \mathbf{b}$$

where the matrix of coefficients $\mathbf{A}_\alpha = \begin{pmatrix} 2 & -1 & 4 \\ \alpha & 1 & -1 \\ -1 & 1 & \alpha \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 13 \\ 5 \\ 5 \end{pmatrix}$.

We find that

$$\det \mathbf{A}_\alpha = \begin{vmatrix} 2 & -1 & 4 \\ \alpha & 1 & -1 \\ -1 & 1 & \alpha \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 1 & \alpha \end{vmatrix} - \alpha\begin{vmatrix} -1 & 4 \\ 1 & \alpha \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix}$$
$$= 2(\alpha + 1) - \alpha(-\alpha - 4) - (1 - 4) = \alpha^2 + 6\alpha + 5.$$

[2 marks]

(iv) According to Cramer's rule, the system of linear equations has a unique solution when

$$\det \mathbf{A}_\alpha \neq 0 \Leftrightarrow \alpha^2 + 6\alpha + 5 \neq 0 \Leftrightarrow (\alpha + 1)(\alpha + 5) \neq 0 \Leftrightarrow \alpha \neq \begin{cases} -1 \\ -5. \end{cases}$$

[2 marks]

(v) (a). When $\alpha = -1$ we have

$$2x - y + 4z = 13, \quad (1)$$
$$-x + y - z = 5, \quad (2)$$
$$-x + y - z = 5. \quad (3)$$

We use Gauss elimination. Adding $2 \times$ Eq. (2) to Eq.(1) and subtracting Eq. (2) from Eq. (3):

$$y + 2z = 23, \quad (4)$$
$$-x + y - z = 5, \quad (5)$$
$$0 = 0. \quad (6)$$

Multiply Eq. (5) by −1 and adding Eq. (4) to Eq. (5) and exchanging Eq. (4) and Eq. (5):

$$x + 3z = 18,$$
$$y + 2z = 23,$$
$$0 = 0.$$

Therefore, the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 23 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \Leftrightarrow \mathbf{r} = \mathbf{r}_0 + \lambda \mathbf{d}, \quad \lambda \in \mathbb{R}.$$

Note to marker. The vectors $\mathbf{r}_0$ and $\mathbf{d}$ are, of course, not unique. Any point $\mathbf{r}_0$ on the line is accepted as a right anser as is any numerical multiple of the directional vector $\mathbf{d}$. The method to find the line-solution is not unique either. Any sounds methods will be acceptable. Therefore, full marks are given for equally valid (alternative) solutions and methods. [3 marks]

(b). We find that

$$\mathbf{A}_{-1}\mathbf{r}_0 = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 18 \\ 23 \\ 0 \end{pmatrix} = \begin{pmatrix} 36 - 23 + 0 \\ -18 + 23 + 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ 5 \end{pmatrix}$$

$$\mathbf{A}_{-1}\mathbf{d} = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 + 2 + 4 \\ 3 - 2 - 1 \\ 3 - 2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, $\mathbf{r}_0$ is a particular solution while $\mathbf{d}$ is a solution to the homogeneous Eq. $\mathbf{A}_{-1}\mathbf{r} = \mathbf{0}$. [2 marks]

(c). The solution represents a straight line through the point $\mathbf{r}_0$ and direction along the vector $\mathbf{d}$. [2 marks]

(d). The normal vector of the second plane is $\mathbf{n}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$. [1 mark]

The cross-product

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \mathbf{k} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

[1 mark]

Planes 2 and 3 are identical. Plane 1 intersect plane 2 in a line as they are not parallel. Hence, $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{d}$ is a directional vector for the line-solution obtained in question (e)(i). [1 mark]

(vi) (a). When $\alpha = -5$ we have

$$2x - y + 4z = 13, \tag{7}$$
$$-5x + y - z = 5, \tag{8}$$
$$-x + y - 5z = 5. \tag{9}$$

We use Gauss elimination. Adding 2× Eq. (9) to Eq. (7) and −5× Eq. (9) to Eq. (8) we find

$$y - 6z = 23, \tag{10}$$
$$-4y + 24z = -20, \tag{11}$$
$$-x + y - 5z = 5. \tag{12}$$

Adding 4× Eq. (10) to Eq. (11) yields 0 = 63 which is clearly false, so the system of equations has no solution. [2 marks]

(b). The three planes in Eqs.(1)-(3) do not intersect. [1 mark]

Classwork 6 – Transforming Areas & Volumes: *Answers*

(a) $\mathbf{r}_2 = \mathbf{T}\mathbf{r}_1 \Leftrightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_1x_1 + b_1y_1 \\ a_2x_1 + b_2y_1 \end{pmatrix}$, that is, $x_2 = a_1x_1 + b_1y_1$; $y_2 = a_2x_1 + b_2y_1$. The origin $x_1 = y_1 = 0$ is transformed into itself, $x_2 = y_2 = 0$.

(b) Just by inspection of the Figure, we find $\mathbf{r}_\text{A} = \begin{pmatrix} u \\ v \end{pmatrix}, \mathbf{r}_\text{B} = \begin{pmatrix} u+s \\ v \end{pmatrix}, \mathbf{r}_\text{C} = \begin{pmatrix} u+s \\ v+s \end{pmatrix}, \mathbf{r}_\text{D} = \begin{pmatrix} u \\ v+s \end{pmatrix}$.

Similarly, by inspection, $\overrightarrow{AB} = \overrightarrow{DC} = s\mathbf{i}, \overrightarrow{AD} = \overrightarrow{BC} = s\mathbf{j}$.

(c) Consider a line $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}$. Since the transformation is linear, we find that $\mathbf{T}\mathbf{r} = \mathbf{T}(\mathbf{r}_0 + \lambda\mathbf{d}) = \mathbf{T}\mathbf{r}_0 + \mathbf{T}(\lambda\mathbf{d}) = \mathbf{T}\mathbf{r}_0 + \lambda(\mathbf{T}\mathbf{d})$ which indeed is a straight line passing through the point $\mathbf{T}\mathbf{r}_0$ and direction vector $\mathbf{T}\mathbf{d}$.

(d) We find that the corners of the square transform into

$$\mathbf{r}_E = \mathbf{T}\mathbf{r}_A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_1u + b_1v \\ a_2u + b_2v \end{pmatrix}, \quad \mathbf{r}_F = \mathbf{T}\mathbf{r}_B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u+s \\ v \end{pmatrix} = \begin{pmatrix} a_1(u+s) + b_1v \\ a_2(u+s) + b_2v \end{pmatrix},$$

$$\mathbf{r}_G = \mathbf{T}\mathbf{r}_C = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u+s \\ v+s \end{pmatrix} = \begin{pmatrix} a_1(u+s) + b_1(v+s) \\ a_2(u+s) + b_2(v+s) \end{pmatrix}, \text{ and}$$

$$\mathbf{r}_H = \mathbf{T}\mathbf{r}_D = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u \\ v+s \end{pmatrix} = \begin{pmatrix} a_1u + b_1(v+s) \\ a_2u + b_2(v+s) \end{pmatrix}, \text{ respectively.}$$

Using $\overrightarrow{EF} = \mathbf{r}_F - \mathbf{r}_E, \overrightarrow{HG} = \mathbf{r}_G - \mathbf{r}_H, \overrightarrow{EH} = \mathbf{r}_H - \mathbf{r}_E, \overrightarrow{FG} = \mathbf{r}_G - \mathbf{r}_F$, , we find

$$\overrightarrow{EF} = \overrightarrow{HG} = \begin{pmatrix} a_1s \\ a_2s \end{pmatrix}, \quad \overrightarrow{EH} = \overrightarrow{FG} = \begin{pmatrix} b_1s \\ b_2s \end{pmatrix}.$$

(e) We evaluate the results above using $\mathbf{T} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}, \mathbf{r}_A = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $s = 3$, yielding

$\mathbf{r}_E = \begin{pmatrix} -5 \\ -6 \end{pmatrix}, \mathbf{r}_F = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{r}_G = \begin{pmatrix} 10 \\ 12 \end{pmatrix}, \mathbf{r}_H = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \overrightarrow{EF} = \overrightarrow{HG} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}; \overrightarrow{EH} = \overrightarrow{FG} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$. See next pg.

(f) (i) Since $\overrightarrow{EF} = a_1s\mathbf{i} + a_2s\mathbf{j}$ and $\overrightarrow{EH} = b_1s\mathbf{i} + b_2s\mathbf{j}$, the area of the parallelogram is

$$\left|\overrightarrow{EF} \times \overrightarrow{EH}\right| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1s & a_2s & 0 \\ b_1s & b_2s & 0 \end{matrix}\right\| = |a_1sb_2s - b_1sa_2s| = s^2\,|(a_1b_2 - a_2b_1)| = s^2\,|\det\mathbf{T}|.$$

Hence, since the original area was s^2, the area scale factor is multiplied by $|\det\mathbf{T}|$. (Note that, in the equation for the area, the outer bars signify the absolute value while the inner bars signify the determinant.) (ii) Inserting the values, we find $|\det\mathbf{T}| = |12 - 4| = 8$.

(g) Yes, because any shape can be considered to be an assembly of small squares.

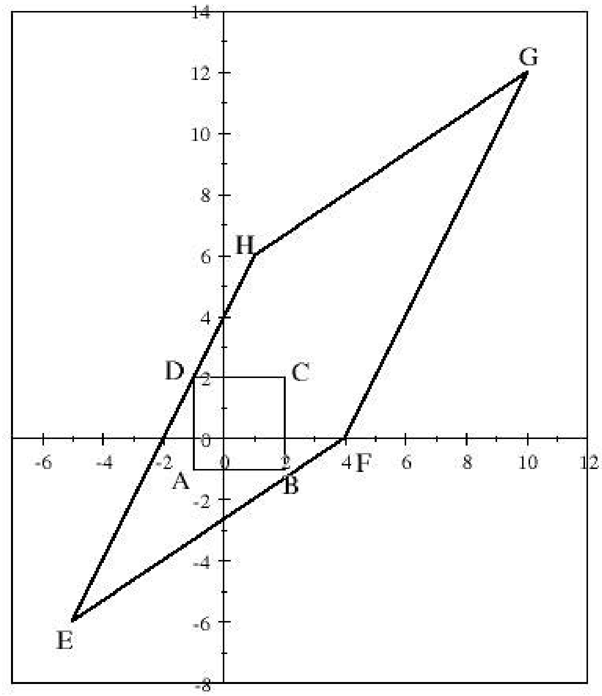
(h) (i) The transformations of the natural basis vectors $f(\mathbf{e}_j)$ are the column vectors $\mathbf{a}_j$ of the matrix defining the transformation:

$$\mathbf{a}_1 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{a}_2 = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \mathbf{a}_3 = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

(ii) The volume is $\left|\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)\right| = \left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \right\| = \left|\det \mathbf{T}\right|$ as before.

(iii) Yes, because any solid can be considered to be an assembly of small cubes.

(e) Sketch of square *ABCD* and its transform, the parallelogram *EFGH:*



Classwork 7 – Discover the Orthogonal Matrix Answers

(a) (i) The magnitude of $\mathbf{u}$ is the root of the sum of the squares of the components, that is, $|\mathbf{u}| = \sqrt{u_x^2 + u_y^2} = 1 \Leftrightarrow u_x^2 + u_y^2 = 1$. Identical argument for $\mathbf{v}$ yields $v_x^2 + v_y^2 = 1$.

The two vectors are orthogonal if $\mathbf{u}\bullet\mathbf{v} = 0 \Leftrightarrow \begin{pmatrix} u_x \\ u_y \end{pmatrix} \bullet \begin{pmatrix} v_x \\ v_y \end{pmatrix} = u_x v_x + u_y v_y = 0$.

(ii) In matrix form, the conditions are $u_x^2 + u_y^2 = \begin{pmatrix} u_x & u_y \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathbf{u}^t\mathbf{u} = \mathbf{v}^t\mathbf{v} = 1$ and

$u_x v_x + u_y v_y = \begin{pmatrix} u_x & u_y \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x & v_y \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathbf{u}^t\mathbf{v} = \mathbf{v}^t\mathbf{u} = 0$, respectively.

(b) Since $|\mathbf{a}| = \sqrt{3^2 + (-4)^2} = 5$, $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$, so $\hat{\mathbf{b}}_1 = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$ and $\hat{\mathbf{b}}_2 = -\hat{\mathbf{b}}_1 = \begin{pmatrix} -4/5 \\ -3/5 \end{pmatrix}$.

We easily check that $\hat{\mathbf{a}}\bullet\hat{\mathbf{b}}_1 = \hat{\mathbf{a}}\bullet\hat{\mathbf{b}}_2 = 0$ and (clearly) $|\hat{\mathbf{b}}_i|^2 = \hat{\mathbf{b}}_i\bullet\hat{\mathbf{b}}_i = 1,\ i = 1, 2$.

(c) We find that $\mathbf{O}^t\mathbf{O} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} u_x^2 + u_y^2 & u_x v_x + u_y v_y \\ v_x u_x + v_y u_y & v_x^2 + v_y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ using the conditions in (a).

(d) (i) We find $\mathbf{q} = \mathbf{A}\mathbf{p} \Leftrightarrow \begin{pmatrix} q_x \\ q_y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a_{11}p_x + a_{12}p_y \\ a_{21}p_x + a_{22}p_y \end{pmatrix}$. If $|\mathbf{q}| = |\mathbf{p}|$, then $|\mathbf{q}|^2 = |\mathbf{p}|^2$. Therefore,

$$\begin{aligned} q_x^2 + q_y^2 &= \left(a_{11}p_x + a_{12}p_y\right)^2 + \left(a_{21}p_x + a_{22}p_y\right)^2 \\ &= (a_{11}^2 + a_{21}^2)p_x^2 + (a_{12}^2 + a_{22}^2)p_y^2 + 2(a_{11}a_{12} + a_{21}a_{22})p_x p_y \,. \\ &= p_x^2 + p_y^2 \end{aligned}$$

The conditions for this to be true are $a_{11}^2 + a_{21}^2 = a_{12}^2 + a_{22}^2 = 1$ and $a_{11}a_{12} + a_{21}a_{22} = 0$, which are exactly the same as the conditions on the elements of $\mathbf{O}$ in part (c).

We have show that, if $\mathbf{q} = \mathbf{A}\mathbf{p}$ and $|\mathbf{q}| = |\mathbf{p}|$ then $\mathbf{A}$ is an orthogonal matrix.

(ii) Since $\mathbf{p}^t = \begin{pmatrix} p_x & p_y \end{pmatrix}$ is an 1×2 matrix and $\mathbf{A}^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$ is an 2×2 matrix, only the matrix product in (2) is well-defined and indeed

$\begin{pmatrix} q_x & q_y \end{pmatrix} = \begin{pmatrix} p_x & p_y \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \Leftrightarrow \mathbf{q}^t = \mathbf{p}^t\mathbf{A}^t$.

(e) $\mathbf{q}_1 \bullet \mathbf{q}_2 = \mathbf{q}_1^t \mathbf{q}_2 = (\mathbf{A}\mathbf{p}_1)^t \mathbf{A}\mathbf{p}_2 = \mathbf{p}_1^t \mathbf{A}^t \mathbf{A}\mathbf{p}_2 = \mathbf{p}_1^t \left(\mathbf{A}^t\mathbf{A}\right)\mathbf{p}_2$, using the result of part (d)(ii).

$\mathbf{p}_1 \bullet \mathbf{p}_2 = \mathbf{p}_1^t \mathbf{p}_2 = \mathbf{p}_1^t \mathbf{I} \mathbf{p}_2$.

Therefore, $\mathbf{q}_1 \bullet \mathbf{q}_2 = \mathbf{p}_1 \bullet \mathbf{p}_2$ if and only if $\mathbf{A}^t\mathbf{A} = \mathbf{I}$.

We have shown that if $\mathbf{q}_i = \mathbf{A}\mathbf{p}_i, i = 1,2$ and $\mathbf{q}_1 \bullet \mathbf{q}_2 = \mathbf{p}_1 \bullet \mathbf{p}_2$ then $\mathbf{A}$ is an orthogonal matrix.

(f) Yes. The column vectors are normalised since $\cos^2\theta + \sin^2\theta = 1$ and they are orthogonal since $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \bullet \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$. Also, we can evaluate directly

$$\mathbf{R}_\theta^t \mathbf{R}_\theta = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$
$$= \begin{pmatrix}\cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta\end{pmatrix}$$
$$= \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.$$

(g) $\mathbf{O}_1 = \begin{pmatrix}3/5 & 4/5 \\ -4/5 & 3/5\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ with $\theta = -53.13^\circ$ which represents an a *clockwise* rotation of 53.13° of the plane about the origin.

$\mathbf{O}_2 = \begin{pmatrix}3/5 & -4/5 \\ -4/5 & -3/5\end{pmatrix} = \begin{pmatrix}3/5 & 4/5 \\ -4/5 & 3/5\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} = \mathbf{O}_1\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$ which represents a reflection in the *x*-axis $(y \to -y)$ followed by a 53.13° clockwise rotation.

(h) $\mathbf{t}_1 = \mathbf{O}_1\mathbf{s} = \begin{pmatrix}3/5 & 4/5 \\ -4/5 & 3/5\end{pmatrix}\begin{pmatrix}-5 \\ 7\end{pmatrix} = \begin{pmatrix}13/5 \\ 41/5\end{pmatrix}$ and $\mathbf{t}_2 = \mathbf{O}_2\mathbf{s} = \begin{pmatrix}3/5 & -4/5 \\ -4/5 & -3/5\end{pmatrix}\begin{pmatrix}-5 \\ 7\end{pmatrix} = \begin{pmatrix}-43/5 \\ -1/5\end{pmatrix}$.

We find $|\mathbf{s}| = \sqrt{(-5)^2 + 7^2} = |\mathbf{t}_1| = \sqrt{(13/5)^2 + (41/5)^2} = |\mathbf{t}_2| = \sqrt{(-43/5)^2 + (-1/5)^2} = \sqrt{74}$, that is, that magnitude of all three vectors is $\sqrt{74}$.

We evaluate the dot-product: $\hat{\mathbf{s}} \bullet \hat{\mathbf{t}}_1 = \dfrac{\mathbf{s}\bullet\mathbf{t}_1}{|\mathbf{s}||\mathbf{t}_1|} = \dfrac{\mathbf{s}\bullet\mathbf{t}_1}{74} = \dfrac{-65/5 + 287/5}{74} = 0.6 = \cos\theta_{\mathbf{s},\mathbf{t}_1}$ which yields an angle between $\mathbf{s}$ and $\mathbf{t}_1$ $\theta_{\mathbf{s},\mathbf{t}_1} = 53.13^\circ$. Similarly, we find that

$\hat{\mathbf{s}} \bullet \hat{\mathbf{t}}_2 = \dfrac{\mathbf{s}\bullet\mathbf{t}_2}{|\mathbf{s}||\mathbf{t}_2|} = \dfrac{\mathbf{s}\bullet\mathbf{t}_2}{74} = \dfrac{215/5 - 7/5}{74} = 0.5622 = \cos\theta_{\mathbf{s},\mathbf{t}_2}$ which corresponds to an angle between $\mathbf{s}$ and $\mathbf{t}_2$ of $\theta_{\mathbf{s},\mathbf{t}_2} = 55.79^\circ$.

Relative to the positive *x*-axis, the vector **s** lies at +125.54° (anti-clockwise), the vector $\mathbf{t}_1$ lies at +72.41° (anti-clockwise), and the vector $\mathbf{t}_2$ lies at +181.33° (anti-clockwise) or -178.67° (clockwise). The latter can be obtained by reflection of **s** in the *x*-axis to produce $\begin{pmatrix}-5 \\ -7\end{pmatrix}$ followed by a 53.13° clockwise rotation. Please draw the position vectors in a diagram yourself and I will then save part of a tree by not having to copy another page ☺!

Classwork 8 - Eigenvalue Problem: Answers

1. (i) We find the eigenvalues by solving the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$.

$$\det(\mathbf{A}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}=0 \Leftrightarrow (4-\lambda)(1-\lambda)-4=0 \Leftrightarrow \lambda^2-5\lambda=\lambda(\lambda-5)=0$$

so therefore, the two eigenvalues are $\lambda_1=5$ and $\lambda_2=0$.

(ii) We find the determinant of **A** and its trace: $\det\mathbf{A}=4-4=0,$ Trace $\mathbf{A}=4+1=5$ so we have that the determinant equals the product of the two eigenvalues while the trace equals the sum of the two eigenvalues.

(iii) To find the associated eigenvectors, we must solve the homogeneous equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for each eigenvalue separately.

$$\lambda_1=5: \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x_1+2y_1=0 \\ 2x_1-4y_1=0 \end{cases} \Leftrightarrow x_1=2y_1 \text{ so an eigenvector}$$

associated with the eigenvalue $\lambda_1=5$ is $\mathbf{x}_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (or a numerical multiple hereof).

$$\lambda_2=0: \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 4x_2+2y_2=0 \\ 2x_2+y_2=0 \end{cases} \Leftrightarrow y_2=-2x_2 \text{ so an eigenvector}$$

associated with the eigenvalue $\lambda_2=0$ is $\mathbf{x}_2=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (or a numerical multiple hereof).

(iv) The scalar product $\mathbf{x}_1 \bullet \mathbf{x}_2=-2+2=0$ and hence the eigenvectors are orthogonal (perpendicular). This is guaranteed by **A** being a real and symmetric matrix $\mathbf{A}^t=\mathbf{A}$.

(v) We construct the matrix of eigenvectors $\mathbf{S}=(\mathbf{x}_1\ \mathbf{x}_2)=\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$. We notice that the determinant of the matrix of eigenvectors $\det\mathbf{S}=4+1=5\neq 0$ so the matrix is invertible. We find $\mathbf{S}^{-1}=\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ and therefore

$$\mathbf{S}^{-1}\mathbf{AS}=\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}=\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 5 & 0 \end{pmatrix}=\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

2. To find the eigenvalues, we solve the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ and to find the associated eigenvectors, we solve the associated homogeneous equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for each λ.

(i) $\det(\mathbf{A}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix}=0 \Leftrightarrow (5-\lambda)(2-\lambda)-4=0 \Leftrightarrow \lambda^2-7\lambda+6=0$ so

$\lambda=\dfrac{7\pm\sqrt{7^2-4\cdot1\cdot6}}{2}=\dfrac{7\pm5}{2}=\begin{cases} 6 \\ 1 \end{cases}$. Notice that $\lambda_1\lambda_2=6=\det\mathbf{A},$ $\lambda_1+\lambda_2=7=$Trace $\mathbf{A}$ as they must be.

We find the respective eigenvectors:

$\lambda_1 = 6$: $\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x_1 - 2y_1 = 0 \\ -2x_1 - 4y_1 = 0 \end{cases} \Leftrightarrow x_1 = -2y_1$ so an eigenvector associated with the eigenvalue $\lambda_1 = 6$ is $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Divide by $\sqrt{5}$ to normalise.

$\lambda_2 = 1$: $\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 4x_2 - 2y_2 = 0 \\ -2x_2 + y_2 = 0 \end{cases} \Leftrightarrow y_2 = 2x_2$ so an eigenvector associated with the eigenvalue $\lambda_2 = 1$ is $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Divide by $\sqrt{5}$ to normalise.

The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal, $\mathbf{x}_1 \bullet \mathbf{x}_2 = 0$, because **A** is a real and symmetric matrix $\mathbf{A}^t = \mathbf{A}$.

(ii) $\det(\mathbf{B} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} 5-\lambda & -7 \\ 1 & -3-\lambda \end{vmatrix} = 0 \Leftrightarrow (5-\lambda)(-3-\lambda) + 7 = 0 \Leftrightarrow \lambda^2 - 2\lambda - 8 = 0$

so $\lambda = \dfrac{2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-8)}}{2} = \dfrac{2 \pm 6}{2} = \begin{cases} 4 \\ -2 \end{cases}$. Note that $\lambda_1\lambda_2 = -8 = \det \mathbf{B}$ and $\lambda_1 + \lambda_2 = 2 = \text{Trace}\ \mathbf{B}$ as they must be.

We find the respective eigenvectors:

$\lambda_1 = 4$: $\begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 - 7y_1 = 0 \\ x_1 - 7y_1 = 0 \end{cases} \Leftrightarrow x_1 = 7y_1$ so an eigenvector associated with the eigenvalue $\lambda_1 = 4$ is $\mathbf{x}_1 = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$. Divide by $\sqrt{7^2 + 1^2} = \sqrt{50}$ to normalise.

$\lambda_2 = -2$: $\begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 7x_2 - 7y_2 = 0 \\ x_2 - y_2 = 0 \end{cases} \Leftrightarrow x_2 = y_2$ so an eigenvector associated with the eigenvalue $\lambda_2 = -2$ is $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Divide by $\sqrt{1^2 + 1^2} = \sqrt{2}$ to normalise. Note that the eigenvectors are *not* orthogonal in this case.

(iii) $\hat{\mathbf{x}}_1 = \mathbf{x}_1 / |\mathbf{x}_1| = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}, \hat{\mathbf{x}}_2 = \mathbf{x}_2 / |\mathbf{x}_2| = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

(iv) The matrix of normalised eigenvectors: $\mathbf{S} = (\hat{\mathbf{x}}_1\ \hat{\mathbf{x}}_2) = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2\sqrt{\sqrt{5}} \end{pmatrix}$.The transpose matrix $\mathbf{S}^t = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$ and we note

$\mathbf{S}^t\mathbf{S} = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(v) We know that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \Lambda \Leftrightarrow \mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1}$. Hence, we find

$$\mathbf{A}^k = (\mathbf{S}\Lambda\mathbf{S}^{-1})(\mathbf{S}\Lambda\mathbf{S}^{-1})\cdots(\mathbf{S}\Lambda\mathbf{S}^{-1}) \text{ with } k \text{ factors and since } \mathbf{S}\mathbf{S}^{-1} = \mathbf{I}$$

$$= \mathbf{S}\Lambda^k\mathbf{S}^{-1}$$

$$= \mathbf{S}\begin{pmatrix} \lambda_1^k & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}^{-1} \qquad \text{since } \lambda_2 = 1$$

$$= \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1^k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2\lambda_1^k & \lambda_1^k \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 4\lambda_1^k + 1 & -2\lambda_1^k + 2 \\ -2\lambda_1^k + 2 & \lambda_1^k + 4 \end{pmatrix} \qquad \text{Now insert } k = 247.$$

3. Let **x** be an eigenvector with eigenvalue λ for the matrix **B.** Then we find $\mathbf{B}^2\mathbf{x} = \mathbf{B}(\mathbf{B}\mathbf{x}) = \mathbf{B}(\lambda\mathbf{x}) = \lambda(\mathbf{B}\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$. The eigenvector **x** for the matrix **B** is therefore also an eigenvector for the matrix $\mathbf{B}^2$ but with eigenvalue λ^2. Since the eigenvlues for **B** are $\lambda_1 = 4$ and $\lambda_2 = -2$, the eigenvalues for $\mathbf{B}^2$ are $\lambda = \begin{cases} 16 \\ 4 \end{cases}$.

4. (i) Consider a general diagonal matrix **A** with diagonal elements a_{11}, a_{22} and a_{33}. The characteristic equations reads

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} a_{11} - \lambda & 0 & 0 \\ 0 & a_{22} - \lambda & 0 \\ 0 & 0 & a_{33} - \lambda \end{vmatrix} = 0 \Leftrightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0 \text{ so}$$

the eigenvalues are the elements in the diagonal matrix $\lambda_1 = a_{11}, \lambda_2 = a_{22,}$ and $\lambda_3 = a_{33}$. Note that $\lambda_1\lambda_2\lambda_3 = a_{11}a_{22}a_{33} = \det\mathbf{A}$ & $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = \text{Trace }\mathbf{A}$ as they must be.

The eigenvector associated with $\lambda_1 = a_{11}$ we find by solving the associated homogenous equation

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} - \lambda & 0 \\ 0 & 0 & a_{33} - \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 0 = 0 \\ (a_{22} - \lambda)y_1 = 0 \Leftrightarrow y_1 = z_1 = 0 \text{ so} \\ (a_{33} - \lambda)z_1 = 0 \end{cases}$$

$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = a_{11}$. Likewise we would

find that $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors associated with $\lambda_2 = a_{22}$ and $\lambda_3 = a_{33}$,

respectively. Hence, the eigenvalues and eigenvectors for the matrix **A** given in the question are the diagonal elements, that is, are

$\lambda_1 = 3$ associated with $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda_2 = 5$ associated with $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\lambda_3 = 27$ associated with $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(ii) The characteristic equation

$$\det(\mathbf{B} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Leftrightarrow -\lambda \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 2-\lambda \\ 1 & 0 \end{vmatrix} = (\lambda^2 - 1)(2-\lambda) = 0$$

so the eigenvalues are $\lambda_1 = 2, \lambda_2 = 1$ and $\lambda_3 = -1$. Note that $\lambda_1\lambda_2\lambda_3 = -2 = \det \mathbf{B}$ & $\lambda_1 + \lambda_2 + \lambda_3 = 2 = \text{Trace } \mathbf{B}$ as they must be.

The eigenvector associated with $\lambda_1 = 2$ we find by solving the associated homogenous equation $\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2x_1 + z_1 = 0 \\ 0 = 0 \\ x_1 - 2z_1 = 0 \end{cases} \Leftrightarrow x_1 = z_1 = 0$ so $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

The eigenvector associated with $\lambda_2 = 1$ we find by solving the associated homogenous equation $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x_2 + z_2 = 0 \\ y_2 = 0 \\ x_2 - z_2 = 0 \end{cases} \Leftrightarrow x_2 = z_2, y_2 = 0$ so $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

The eigenvector associated with $\lambda_3 = -1$ we find by solving the associated homogenous equation $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_3 + z_3 = 0 \\ y_3 = 0 \\ x_3 + z_3 = 0 \end{cases} \Leftrightarrow x_3 = -z_3, y_3 = 0$ so $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector.

(iii) The characteristic equation

$$\det(\mathbf{C} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 2-\lambda \\ 2 & 0 \end{vmatrix} = 0$$

Hence $(2-\lambda)(\lambda^2 - \lambda - 6) = 0$ so the eigenvalues are $\lambda_1 = 3, \lambda_2 = 2$ and $\lambda_3 = -2$. Note that $\lambda_1\lambda_2\lambda_3 = -12 = \det \mathbf{C}$ & $\lambda_1 + \lambda_2 + \lambda_3 = 3 = \text{Trace } \mathbf{C}$ as they must be.

The eigenvectors are $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda_1 = 3$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for $\lambda_2 = 2$, and $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ for $\lambda_3 = -2$.

(iv) The characteristic equation $\det\left(\mathbf{D}-\lambda\mathbf{I}\right)=0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$. Evaluating the determinant, we find $\lambda^3 - 4\lambda^2 - \lambda + 4 = (\lambda-4)(\lambda-1)(\lambda+1) = 0$, so the three eigenvalues are $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = -1$. Note that $\lambda_1\lambda_2\lambda_3 = -4 = \det\mathbf{D}$ & $\lambda_1+\lambda_2+\lambda_3=4=\text{Trace }\mathbf{D}$ as they must be.

To find the associated eigenvectors we solve the homogeneous equations:

$$\lambda_1 = 4: \begin{pmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1 + 2y_1 + z_1 = 0 \\ 2x_1 - 3y_1 + z_1 = 0 \\ x_1 + y_1 - 2z_1 = 0 \end{cases} \Leftrightarrow \{x_1 = y_1 = z_1 .$$

Therefore, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = 4.$.

$$\lambda_2 = 1: \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2y_2 + z_2 = 0 \\ 2x_2 + z_2 = 0 \\ x_2 + y_2 + z_2 = 0 \end{cases} \Leftrightarrow \{x_2 = y_2 = -\tfrac{1}{2}z_2 .$$

Hence, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 1$.

$$\lambda_3 = -1: \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x_3 + 2y_3 + z_3 = 0 \\ 2x_3 + 2y_3 + z_3 = 0 \\ x_3 + y_3 + 3z_3 = 0 \end{cases} \Leftrightarrow \begin{cases} y_3 = -x_3 \\ z_3 = 0 \end{cases} .$$

Therefore, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_3 = -1$.