Supplement to Sec. 6.2 in "Mathematics: Linear Algebra - Lecture Notes."

To determine the angle θ between the two normal vectors, we apply definition 3.3:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \quad 0 \le \theta \le \pi,$$

where $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are the unit normal vectors to the planes, respectively. Then we identify

$$\theta_p = \begin{cases} \theta & \text{if } 0 \le \theta \le \pi/2, \\ \pi - \theta & \text{if } \pi/2 \le \theta \le \pi. \end{cases}$$

Note that when the angle $\theta \in [\pi/2, \pi]$ between the two normal vectors, the (acute) angle between the planes $\theta_p = \pi - \theta$, see Fig. 6.3(b).

Clearly, when $0 \le \theta \le \pi/2$, $\cos\theta = \cos\theta_p$. Also note that the scalar product $\mathbf{n}_1 \cdot \mathbf{n}_2 \ge 0$. However, when $\pi/2 \le \theta \le \pi$, $\cos\theta = \cos(\pi - \theta_p) = -\cos\theta_p$ and we note that the scalar product $\mathbf{n}_1 \cdot \mathbf{n}_2 \le 0$. Therefore, we can conclude that the acute angle θ_p between two planes satisfies:

$$\cos\theta_p = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} \quad 0 \le \theta_p \le \pi/2.$$

Procedure to determine the (acute) angle θ_p between two planes in $\mathbb{R}^3$

1. The acute angle $\theta_p \in [0, \pi/2]$ between the planes is the solution to the equation

   $$\cos\theta_p = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|},$$

   where $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors to the two planes, respectively.

Example 6.2. Consider the two planes from Ex. 6.1. The magnitude of the normal vectors are

$$\mathbf{n}_1 = (5, -4, -3) \Rightarrow |\mathbf{n}_1| = \sqrt{5^2 + (-4)^2 + (-3)^2} = \sqrt{50},$$
$$\mathbf{n}_2 = (-2, 1, 1) \Rightarrow |\mathbf{n}_2| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}.$$

The dot-product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (5, -4, -3) \cdot (-2, 1, 1) = -10 - 4 - 3 = -17.$$

Hence the acute angle θ_p between the two planes satisfies:

$$\cos\theta_p = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{17}{\sqrt{50}\sqrt{6}} \Rightarrow \theta_p \approx 0.193 \text{ rad} = 11.0^\circ.$$

Updated 26.10.2010

Corrections to "Mathematics: Linear Algebra - Lecture Notes."

Page 15, Sec. 2.9 Summary:
Second entry: Replace $\overline{\mathrm{AB}}$ with $\overrightarrow{\mathrm{AB}}$.

Page 35, line 2 after Eq. (5.19):
Replace $\mathbf{r} = (x, y, x)$ with $\mathbf{r} = (x, y, z)$

Page 37, line 2:
Replace "right-hand side of Eq. (5.18)" with "right-hand side of Eq. (5.19)"

Corrections to Problem Sheets.

PL6 26/10/2010, Question 3:
Replace: "You may use condition **F** on Fact Sheet 4" with "You may use that two lines intersect if the minimum distance between them is zero, see Sec. 6.5 in the notes."