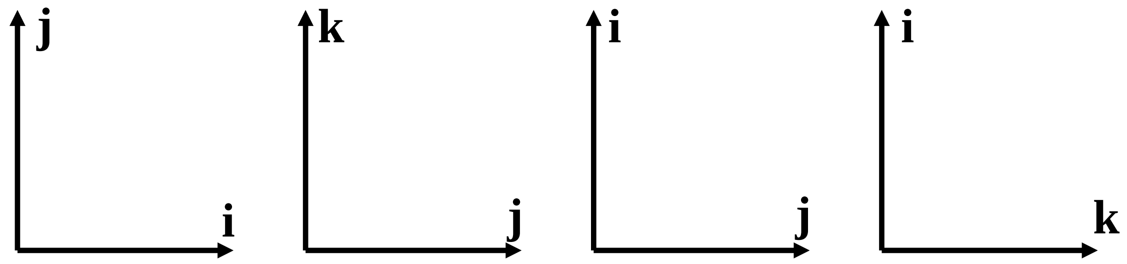
Problems for Lecture 3: Vectors

In the following, consider the three-dimensional space $\mathbb{R}^3$.

1. The unit vectors **i**, **j**, and **k** form a right-handed set. In each of the following pictures showing two of these unit vectors, determine whether the third vector points into or out of the paper.

(a) (b) (c) (d)



The vectors in question 2 to 5 are defined as $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = 7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{C} = 4\mathbf{i} + 5\mathbf{k}$.

2. Find the magnitudes of **A**, **B**, and **C**.

3. Find the unit vectors in the direction of

   (a) **A** (b) **C** (c) **A** + **B**

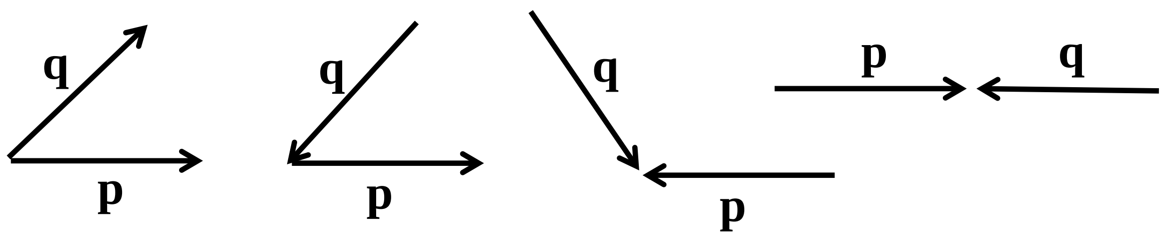
4. Find the following scalar products

   (a) **A•B** (b) **B•C** (c) **C•A**

5. From the answers to question 4, deduce the angle between the vectors in each case.

6. In the following cases determine if $\mathbf{p} \times \mathbf{q}$ is directed into or out of the paper.

(a) (b) (c) (d)



Problems for Lecture 4: Vectors I

1. Consider the vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = 7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{C} = 4\mathbf{i} + 5\mathbf{k}$. Find the following vector (cross) products

   (a) $\mathbf{A} \times \mathbf{B}$ (b) $\mathbf{C} \times \mathbf{B}$ (c) $\mathbf{A} \times \mathbf{C}$

2. (Partly from exam May 2005).

   In the following, consider the three-dimensional space $\mathbb{R}^3$.

   (i) If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, find the angle θ between these two vectors.

   (ii) Find the vector product $\mathbf{a} \times \mathbf{b}$ and its magnitude $|\mathbf{a} \times \mathbf{b}|$.

   (iii) Explain the geometrical significance of $|\mathbf{a} \times \mathbf{b}|$ in relation to a geometrical object defined by $\mathbf{a}$ and $\mathbf{b}$.

   (iv) If $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + \alpha\mathbf{k}$, where $\alpha \in \mathbb{R}$, find $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ in the case where $\alpha = 2$.

   (v) For what value of α is $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = 0$? What can you say about the vectors $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ in this case?

Problems for Lecture 5: Geometry

In questions 1-3, we consider the two-dimensional space $\mathbb{R}^2$.

1. Write down the parametric vector equation of a straight line through the two points $\mathbf{r}_1 = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{r}_2 = 8\mathbf{i} - 5\mathbf{j}$.

   Show that the equation can be written in the form $\dfrac{x-3}{5} = -\dfrac{y-4}{9}$.

2. Write down the parametric vector equation of a straight line of gradient 3 with an intercept on the y-axis at $y = -2$. Obtain the Cartesian (x-y) form as well.

3. For the lines of questions 1 and 2, find

   (a) the direction ratios,

   (b) the direction cosines,

   (c) the unit normal vectors,

   (d) the angle between the two lines,

   (e) the angle between the two normals,

   (f) the perpendicular distances from the origin.

In questions 4 we consider the three-dimensional space $\mathbb{R}^3$.

4. Show that $\dfrac{x+2}{2} = \dfrac{y+8}{5} = \dfrac{z+5}{3}$ and $\dfrac{x-10}{4} = \dfrac{y-22}{10} = \dfrac{z-13}{6}$ represent the same line.

Problems for Lectures 6: Lines & Planes

1. A plane in $\mathbb{R}^3$ is defined by the equation $5x-4y-3z=10$. Find
   (a) the unit normal vector $\hat{\mathbf{n}}_1$,
   (b) the minimal (shortest, perpendicular) distance, d_O, from the origin to the plane,
   (c) the minimal distance, d_P, from the point $\overrightarrow{OP}=(1,3,5)$ to the plane.

2. Find the minimal distance from the point $\overrightarrow{OP}=(1,-2,0)$ to the line joining the two points $\overrightarrow{OA}=(-2,1,2)$ and $\overrightarrow{OB}=(5,5,5)$.

3. For which value of α do the two lines given by $\mathbf{r}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})+\lambda(3\mathbf{i}+2\mathbf{j}+\mathbf{k}), \lambda\in\mathbb{R}$ and $\mathbf{r}=(\alpha\mathbf{i}+\mathbf{j}+\mathbf{k})+\mu(2\mathbf{i}-3\mathbf{j}-4\mathbf{k}), \mu\in\mathbb{R}$ intersect? [*Hint:* You may use condition **F** on Fact Sheet 4.

4. Consider a plane. You may think of the plane as $\mathbb{R}^2$. Consider three vectors $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^2$ and assume that $\mathbf{a}\neq\mathbf{0},\mathbf{b}\neq\mathbf{0}$ and that $\mathbf{a}$ and $\mathbf{b}$ are not parallel, that is.

   (i) Make a graphical representation of the vectors $\mathbf{a}+\mathbf{b},\mathbf{a}-\mathbf{b}$ and $\mathbf{b}-\mathbf{a}$.

   (ii) Explain qualitatively why there exists real numbers $\lambda,\mu\in\mathbb{R}$ such that $\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$.

   (iii) Given $\mathbf{a}=(1,1)$ and $\mathbf{b}=(-1,2)$ find the real numbers $\lambda,\mu\in\mathbb{R}$ such that $\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$ when $\mathbf{c}=(3,7)$.

Problems for Lecture 7&8: Determinants& Linear equations

1. Evaluate the determinants (a) $\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}$ and (b) $\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$. How are the determinates related?

   Now evaluate each of the following 2×2 determinants, and identify how each is "developed" from the determinants in (a) and (b) above:

   (c) $\begin{vmatrix} 1 & 5 \\ 4 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 8 & 2 \\ 2 & 5 \end{vmatrix}$ (e) $\begin{vmatrix} 4 & 2 \\ 5 & 7 \end{vmatrix}$.

2. Find the solutions to each of the following systems of linear equations:

   (a) $\begin{array}{l} 8x_1 + x_2 + 8x_3 = 12 \\ 6x_1 + 4x_2 + 4x_3 = 8 \\ 5x_1 - x_2 + 6x_3 = 15 \end{array}$ (b) $\begin{array}{l} 4x_1 + 7x_2 - 2x_3 = 10 \\ x_1 - 3x_2 + 2x_3 = 6 \end{array}$ (c) $\begin{array}{l} x_1 + x_2 + x_3 = 1 \\ 2x_1 - 2x_2 + 2x_3 = 0 \\ 4x_1 - 4x_2 - 4x_3 = -1 \end{array}$

Problems for Lecture 9: Determinants

1. Evaluate (a) $\begin{vmatrix} 4 & 1 & 2 \\ 7 & 2 & 0 \\ -2 & 3 & 0 \end{vmatrix}$, (b) $\begin{vmatrix} 3 & 2 & 4 \\ 5 & 4 & 8 \\ 8 & 2 & 9 \end{vmatrix}$, (c) $\begin{vmatrix} 2 & 15 & -37 & 8 & 11 \\ 0 & 1 & 6 & 23 & -32 \\ 0 & 0 & 4 & 12 & -29 \\ 0 & 0 & 0 & 10 & 20 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$.

2. Determine which of the following determinants are zero. For each determinant that is zero, try to identify what characteristic of the determinant "ensures" that it is so.

   (a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 12 \\ -5 & 10 & -15 \end{vmatrix}$, (b) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 0 \end{vmatrix}$, (c) $\begin{vmatrix} 7 & 3 & 2 \\ 6 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}$. (d) $\begin{vmatrix} 0 & 7 & 0 \\ 3 & -5 & 6 \\ 2 & 3 & -4 \end{vmatrix}$.

3 Show that the following determinants are zero:

   (a) $\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$ when $a,b,c,d \neq 0$..

   Both determinants can be shown to be zero by multiplying them out, but the point of this question is rather to obtain the results <u>*by using the general properties of determinants.*</u>

4. Consider once again the determinant of the 3×3 matrix

   .

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33}-a_{32}a_{23}) - a_{21}(a_{12}a_{33}-a_{32}a_{13}) + a_{31}(a_{12}a_{23}-a_{22}a_{13}).$$

   (a) How many terms are there in the sum?
   (b) How many multiplications have to be carried out to find each term?
   (c) How many multiplications have to be carried out to calculate the entire determinant?
   (d) Repeat (a)-(c) for a 4×4 determinant
   (e) Repeat (a)-(c) for an *n*×*n* determinant.
   (f) How many multiplications would be required to evaluate a 25×25 determinant?
   (g) The fastest computer in the world can carry out about 3 Penta-Flops = $3 \cdot 10^{15}$ operations per second. How long would it take it to evaluate a 25×25 determinant in this way?

Problems for Lecture 10

Homogeneous Equations, Triple Products, Linear Independence

1. Determine which of following pairs of homogeneous equations have a non-trivial solution and, for those that do, find the equation of the line that represents the solution.

(a) $\begin{aligned} 3x+5y&=0 \\ 2x+4y&=0, \end{aligned}$ (b) $\begin{aligned} 3x-5y&=0 \\ 7x+2y&=0, \end{aligned}$ (c) $\begin{aligned} 6x+3y&=0 \\ 4x+2y&=0, \end{aligned}$ (d) $\begin{aligned} 1.4x-1.2y&=0 \\ -2.1x+1.8y&=0. \end{aligned}$

2. Which of following sets of homogeneous equations have a non-trivial solution?

(a) $\begin{aligned} 8x+y+8z&=0 \\ 6x+4y+4z&=0 \\ 5x-y+6z&=0, \end{aligned}$ (b) $\begin{aligned} 5p+2q+2r&=0 \\ p-q+4r&=0 \\ 7p+r&=0, \end{aligned}$ (c) $\begin{aligned} 12x_1-16x_2+2x_3+8x_4&=0 \\ -6x_1+6x_2+14x_3-3x_4&=0 \\ 10x_1+10x_2-7x_3-5x_4&=0 \\ 11x_1-18x_2+2x_3+9x_4&=0. \end{aligned}$

3. Let $\mathbf{a} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$, $\mathbf{b} = 7\mathbf{i}-2\mathbf{j}+4\mathbf{k}$, and $\mathbf{c} = 4\mathbf{i}+5\mathbf{k}$.

Find the triple scalar products (a) $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}$, (b) $\mathbf{a}\bullet(\mathbf{c}\times\mathbf{b})$, (c) $\mathbf{b}\bullet(\mathbf{c}\times\mathbf{a})$.

Find the triple vector products (d) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, (e) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

4. Three vectors **a, b,** and **c** are *linearly dependent* if it is possible to write one of the vectors as a linear combination of the other two, for example, $\mathbf{a}=p\mathbf{b}+q\mathbf{c}$, $p,q\in\mathbb{R}$.

Three vectors are *linearly independent* if it is <u>*not*</u> possible to do this.

More generally, we say that a set of vectors $\mathbf{a}_1,\mathbf{a}_{2,}\ldots,\mathbf{a}_n$ is *linearly dependent* if there exists numbers $c_1,c_2,\ldots,c_n$ **not all** equal to zero such that $c_1\mathbf{a}_1+c_2\mathbf{a}_2+\cdots+c_n\mathbf{a}_n=\mathbf{0}$ (which, of course implies that one of the vectors with a non-zero coefficient can be written as a linear combination of (a subset of) the other vectors.

The set of vectors is *linearly independent* if $c_1\mathbf{a}_1+c_2\mathbf{a}_2+\cdots+c_n\mathbf{a}_n=\mathbf{0}$ implies all coefficients are zero, $c_1=c_2=\cdots c_n$.

(a) Show that the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ defined in question 3 are linearly independent.

(b) What does linear dependence or independence imply about the determinant formed from the components of the three vectors?

(c) Redefine $\mathbf{a} = (2+\alpha)\mathbf{i}+\mathbf{j}-3\mathbf{k}$. What value of α makes $\mathbf{a},\mathbf{b},\mathbf{c}$ linearly dependent?

5. Show the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$ for any three vectors **a, b** and **c.**

[Hint: Use the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\bullet\mathbf{c})-\mathbf{c}(\mathbf{a}\bullet\mathbf{b})$ **]**

Problems for Lecture 11: Matrices I

In Q 1-3, the 2×2 matrices $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ are $\mathbf{A}=\begin{pmatrix}2 & 3\\ 4 & 6\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & -1\\ 5 & 4\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}-4 & 2\\ 0 & 1\end{pmatrix}$.

1. Find the matrices given by (a) $3\mathbf{A}$, (b) $\mathbf{A}+\mathbf{B}$, (c) $3\mathbf{B}-2\mathbf{C}$.

2. If a column vector (2×1 matrix) is represented by $\mathbf{r}=\begin{pmatrix}3\\ 4\end{pmatrix}$, find

   (a) $\mathbf{r}_1=\mathbf{A}\mathbf{r}$, (b) $\mathbf{r}_2=\mathbf{B}\mathbf{r}$, (c) $\mathbf{r}_3=\mathbf{r}_1+\mathbf{r}_2$. (d) $\mathbf{r}_4=(\mathbf{A}+\mathbf{B})\mathbf{r}$. Explain the property that implies that $\mathbf{r}_3=\mathbf{r}_4$. Prove that property in general for 2×2 matrices and $\mathbf{r}=\begin{pmatrix}x\\ y\end{pmatrix}$.

3. Find the matrix products:

   (a) $\mathbf{AB}$, (b) $\mathbf{BC}$, (c) $\mathbf{CB}$, (d) $\mathbf{AC}$,

   (e) $(\mathbf{AB})\mathbf{C}$, (f) $\mathbf{A}(\mathbf{BC})$, (g) $(\mathbf{A}+\mathbf{B})\mathbf{C}$, (h) $\mathbf{AC}+\mathbf{BC}$.

4. If the matrices $\mathbf{P}=\begin{pmatrix}2 & 3 & 1 & -4\\ 2 & 1 & 0 & 5\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}2 & 4\\ 1 & -1\\ 3 & -1\end{pmatrix}$, $\mathbf{R}=\begin{pmatrix}2 & 1 & 3\\ 4 & -1 & -2\\ -1 & 0 & 1\end{pmatrix}$, either

   calculate, or discard as meaningless, all nine potential products of two of the matrices.

5. Using brute force calculation, find the inverse $\mathbf{A}^{-1}$ of the 2×2 matrix $\mathbf{A}=\begin{pmatrix}9 & 6\\ 5 & 3\end{pmatrix}$, i.e., a matrix matrix $\mathbf{B}$ that satisfies $\mathbf{AB}=\mathbf{BA}=\mathbf{I}$ where $\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is the identity matrix.

6. If $\mathbf{M}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$, evaluate (a) $3\mathbf{M}$, (b) $\det\mathbf{M}$, (c) $\det(3\mathbf{M})$. Show in general that, if $\mathbf{M}$ is an $n\times n$ matrix, then $\det(r\mathbf{M})=r^n\det\mathbf{M}$, where r is a factor. (Reminder: When a matrix is multiplied by a factor r, all elements of the matrix are multiplied by the factor r, i.e., , if m_{ij} is the ij th element of the matrix $\mathbf{M}$, then the ij th element of the matrix $r\mathbf{M}$ is rm_{ij}.)

7. Find the vector resulting from the counter-clockwise rotation of $\mathbf{r}=\begin{pmatrix}3\\ 4\end{pmatrix}$ in question 2 by 60° and 90°, respectively. We will later see how to do this elegantly using matrices!

Problems for Lecture 12: Matrices

1. Find the matrix products:

(a) $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}$, (b) $\begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}$, (c) $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}$, (d) $\begin{pmatrix}5\\6\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}$,

(e) $\begin{pmatrix}1 & -3 & 5 & -7\end{pmatrix}\begin{pmatrix}8\\6\\4\\2\end{pmatrix}$, (f) $\begin{pmatrix}8\\6\\4\\2\end{pmatrix}\begin{pmatrix}1 & -3 & 5 & -7\end{pmatrix}$.

2. (a) Write down the 2×2 rotation matrices $\mathbf{R}_{+45^\circ}, \mathbf{R}_{+90^\circ}$, representing 45° and 90° *anti-clockwise* (*counter-clockwise)* rotations, respectively.

(b) Show that the product of two 45° *anti-clockwise* rotation matrices is equivalent to a single 90° *anti-clockwise* rotation matrix, that is, $\mathbf{R}_{+45^\circ}\mathbf{R}_{+45^\circ} = \mathbf{R}_{+90^\circ}$.

(c) What does the product $\mathbf{R}_{+90^\circ}\mathbf{R}_{+90^\circ} = \mathbf{R}^2_{+90^\circ}$ of two 90° *anti-clockwise* rotation matrices correspond to? What does the product $\mathbf{R}_{-90^\circ}\mathbf{R}_{-90^\circ} = \mathbf{R}^2_{-90^\circ}$ of two 90° *clockwise* rotation matrices correspond to?

3. (a) Write down the 2×2 matrix $\mathbf{R}_{-|\theta|}$ representing a *clockwise* rotation by an angle of $\theta = \sin^{-1}(4/5)$. Express θ in degrees.

(b) If the *clockwise* rotation in 3(a) is followed by an *anti-clockwise* rotation of 45°, find the matrix $\mathbf{R}_{\mathbf{net}}$ representing the net rotation. From the nature of the matrix, deduce whether the net rotation is clockwise or anti-clockwise.

(c) Show that the order of the two axes rotations in parts (a) and (b) is irrelevant.

(d) Deduce the inverse of the matrix $\mathbf{R}_{\mathbf{net}}$.

4. Evaluate the determinants (a) $\det\mathbf{A} = \begin{vmatrix}4 & 0 & 0 & 0\\6 & -1 & 0 & 0\\5 & 4 & 3 & 0\\3 & 2 & 1 & 2\end{vmatrix}$, (b) $\det\mathbf{B} = \begin{vmatrix}4 & 0 & 1 & 0\\6 & -1 & 0 & 0\\5 & 4 & 3 & 0\\3 & 2 & 1 & 2\end{vmatrix}$.

Problems for Lecture 13
Singular Matrices and Linear Equations

1. Which of the following matrices are non-singular? For each that is, find the inverse:

(a) $\mathbf{A}_1 = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$, (b) $\mathbf{A}_2 = \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$, (c) $\mathbf{A}_3 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$.

2. Which of the following matrices are singular? In those cases where a matrix can be seen <u>*by inspection*</u> to be singular, give your reasoning.

(a) $\mathbf{B}_1 = \begin{pmatrix} 8 & 6 & 3 \\ 5 & 8 & 4 \\ 5 & 4 & 2 \end{pmatrix}$, (b) $\mathbf{B}_2 = \begin{pmatrix} 0 & 7 & 0 \\ 3 & -5 & 6 \\ 5 & 4 & 2 \end{pmatrix}$, (c) $\mathbf{B}_3 = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 4 & -1 \\ -1 & 14 & 1 \end{pmatrix}$,

(d) $\mathbf{B}_4 = \begin{pmatrix} 4 & 4 & 4 \\ 2 & 1 & 2 \\ -1 & -1 & 1 \end{pmatrix}$, (e) $\mathbf{B}_5 = \begin{pmatrix} 3.5 & -7.2 & 2.1 & 4.4 \\ 5.3 & 6.2 & 0 & -6.2 \\ 3.5 & -7.2 & 2.1 & 4.4 \\ 1.7 & 0 & -5.3 & 0 \end{pmatrix}$, (f) $\mathbf{B}_6 = \begin{pmatrix} 3 & -5 & 0 & -1 \\ 2 & 1 & 7 & 4 \\ 0 & 6 & -4 & 2 \end{pmatrix}$.

3. Consider the system of linear equations of 3 equations with 3 unknowns x_1, x_2, x_3:

$$\begin{aligned} -x_1 + 2x_2 + 3x_3 &= k_1 \\ 2x_1 + x_2 - 4x_3 &= k_2 \quad \Leftrightarrow \quad \mathbf{Ax} = \mathbf{k} \\ -x_1 - 2x_2 + x_3 &= k_3 \end{aligned}$$

Write down the matrix of the coefficients $\mathbf{A}$ and find in sequence

(a) the determinant $\det \mathbf{A}$,

(b) the matrix of the cofactors $\mathbf{C}$,

(c) the adjoint matrix $\text{adj}\mathbf{A}$,

(d) the inverse $\mathbf{A}^{-1}$,

(e) the general solution to the system of linear equations.

Check that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the 3×3 identity matrix.

4. Obtain the solution of the equations in question 3 for the following values of $\mathbf{k}$:

(a) $\mathbf{k} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, (b) $\mathbf{k} = \begin{pmatrix} 5 \\ -8 \\ 0 \end{pmatrix}$, (c) $\mathbf{k} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

5. Find the determinant of the matrix $\mathbf{D} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 7 & 5 & 5 \\ -1 & 3 & 2 & -1 \\ 3 & -2 & 5 & 4 \end{pmatrix}$.

Problems for Lectures 14: Lines, Planes, Rotations

1. A plane in $\mathbb{R}^3$ is defined by the equation $5x-4y-3z=10$. Find
   (a) the unit normal vector $\hat{\mathbf{n}}_1$,
   (b) the minimal (shortest, perpendicular) distance, d_O, from the origin to the plane,
   (c) the minimum distance, d_P, from the point $\overrightarrow{OP}=(1,3,5)$ to the plane.

2. Consider the two planes $5x-4y-3z=10$ and $-2x+y+z=2$. Find a normal vector $\mathbf{n}_2$ to the second plane (you found a normal to the first in question 1). Hence, find an equation for the *line of intersection* of the two planes in both vector and Cartesian form.

3. What can you say about the intersection of a third plane defined by $x-2y-z=14$ with the two planes specified in question 2?

4. Find the minimal distance from the point $\overrightarrow{OP}=(1,-2,0)$ to the line joining the two points $\overrightarrow{OA}=(-2,1,2)$ and $\overrightarrow{OB}=(5,5,5)$.

5. For which value of α do the two lines given by $\mathbf{r}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})+\lambda(3\mathbf{i}+2\mathbf{j}+\mathbf{k}), \lambda\in\mathbb{R}$ and $\mathbf{r}=(\alpha\mathbf{i}+\mathbf{j}+\mathbf{k})+\mu(2\mathbf{i}-3\mathbf{j}-4\mathbf{k}), \mu\in\mathbb{R}$ intersect?

6. Consider $\mathbb{R}^2$. (a) Write down the matrix of the transformation defined by $\begin{matrix} x' = 2x+3y \\ y' = x-y \end{matrix}$.
   (b) Write down the equations for the transformation whose matrix is $\mathbf{T}=\begin{pmatrix} 7 & -4 \\ 2 & 0 \end{pmatrix}$.

7. State the transformed position of the point (2, 1) under the following transformations:
   (a) Contraction (shrinkage) of factor 2 in the *y*-direction,
   (b) Extension (enlargement) of factor 3 in the x- and y-direction.
   (c) Reflection in the *x*-axis.

8. The 3×3 matrix for a rotation of angle θ about the z-axis is $\mathbf{R}_\theta^z=\begin{pmatrix} \cos\theta & \mp\sin\theta & 0 \\ \pm\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (a) Which signs apply if a positive θ corresponds to anti-clockwise rotation about the positive *z*-axis in a right-handed coordinate system?

   Find the analogous matrices for

   (b) an anti-clockwise rotation about the positive *x*-axis, $\mathbf{R}_\theta^x$,

   (c) an anti-clockwise rotation about the positive *y*-axis, $\mathbf{R}_\theta^y$.

9. Find the resulting vector if $\mathbf{r}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is rotated about the +*z*-axis by (a) 45° anti-clockwise and (b) 45° clockwise. Check that the magnitude remains invariant.

10. Using the same sign-convention as in question 8, the vector in the previous question is rotated first by 45° anti-clockwise about the +*y*-axis and then by 45° clockwise about the +*x*-axis. Find the new vector. Check, once again, that the operation preserves the magnitude of the vector.

Problems for Lecture 15: Eigenvalues & Orthogonal Matrices

1. Find the eigenvalues and eigenvectors of the matrices

   (i) $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ (ii) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. Show that an extension by a factor 2 along a line at 45° to the x and y axes (i.e., parallel to the direction vector $\mathbf{i}+\mathbf{j}$) is represented by the matrix transformation $\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$.

3. The matrix $\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ transforms the vector $\mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$ into the vector $\mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$.

   If the magnitude of the two vectors are the same $|\mathbf{p}| = |\mathbf{q}|$, what conditions between the elements $t_{11}, t_{12}, t_{21}, t_{22}$ must be satisfied?

   Hence show that, under these conditions, the inverse of $\mathbf{T}$ is its transpose, that is, $\mathbf{T}^{-1} = \mathbf{T}^t$. [*HINT:* See Classwork 7, question (d).]

4. Matrices that satisfy the conditions of question 2 are called *orthogonal matrices* (see Classwork 7 for definition). Can all 2×2 orthogonal matrices be represented by rotation matrices of the form $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$?

5. Determine whether the following 2×2 matrices are orthogonal:

   (a) $\mathbf{T}_1 = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$, (b) $\mathbf{T}_2 = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$, (c) $\mathbf{T}_3 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

   Consider your conclusions in the light of questions 2 and 3.

6. One of the following 3×3 matrices is orthogonal, and the other could be made so by a single sign change. Which is which, and what sign change would be necessary?

   (a) $\mathbf{A}_1 = \begin{pmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ (b) $\mathbf{A}_2 = \begin{pmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & -4/5 \end{pmatrix}$.

7. If $\mathbf{O}$ is an orthogonal matrix, show that $\det\mathbf{O} = \pm 1$.

   [*HINT*: Use the fact that $\det(\mathbf{AB}) = (\det\mathbf{A})\cdot(\det\mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are square $n \times n$ matrices. Remember that the inverse of an orthogonal matrix $\mathbf{O}$ is equal to its transpose, $\mathbf{O}^{-1} = \mathbf{O}^t$. One other property of determinants is also needed, see p. 64 in Lecture Notes.]

Problems for Lecture 3: Answers

1. Using the rule stated in Sec. 1.3 of the lecture note, we deduce the following answers:
(a) OUT (b) OUT (c) IN (d) OUT

2. Generally for a three-dimensiona vector, its magnitude (length) is given by $|(x,y,z)| = \sqrt{x^2+y^2+z^2}$. Hence we find $|\mathbf{A}| = \sqrt{2^2+1^1+(-3)^2} = \sqrt{14}$,
$|\mathbf{B}| = \sqrt{7^2+(-2)^2+4^2} = \sqrt{69}$, and $|\mathbf{C}| = \sqrt{4^2+0^2+5^2} = \sqrt{41}$.

3. To find a unit vector $\hat{\mathbf{x}}$ in a given direction $\mathbf{x}$ we simple divide the vector $\mathbf{x}$ by its length, that is, $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Using this procedure, we find

(a) $\hat{\mathbf{a}} = \frac{\mathbf{A}}{\sqrt{14}} = \frac{2}{\sqrt{14}}\mathbf{i} + \frac{1}{\sqrt{14}}\mathbf{j} - \frac{3}{\sqrt{14}}\mathbf{k}$

(b) $\hat{\mathbf{c}} = \frac{\mathbf{C}}{\sqrt{41}} = \frac{4}{\sqrt{41}}\mathbf{i} + \frac{5}{\sqrt{41}}\mathbf{k}$

(c) If $\mathbf{D} = \mathbf{A} + \mathbf{B} = 9\mathbf{i} - \mathbf{j} + \mathbf{k}$, then $\hat{\mathbf{d}} = \frac{\mathbf{D}}{\sqrt{83}} = \frac{9}{\sqrt{83}}\mathbf{i} - \frac{1}{\sqrt{83}}\mathbf{j} + \frac{1}{\sqrt{83}}\mathbf{k}$.

4. We use the definition of a scalar product $\mathbf{A}\bullet\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$ and find:

(a) $\mathbf{A}\bullet\mathbf{B} = 14 - 2 - 12 = 0$ (b) $\mathbf{B}\bullet\mathbf{C} = 28 + 0 + 20 = 48.$ (c) $\mathbf{C}\bullet\mathbf{A} = 8 + 0 - 15 = -7.$

5. We use the definition of the angle θ between two vectors: $\cos\theta = \frac{\mathbf{x}\bullet\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$. Hence

(a) $\cos\theta = \frac{0}{\sqrt{14}\sqrt{69}} = 0 \Rightarrow \theta = \pi/2 = 90^\circ$ (b) $\cos\theta = \frac{48}{\sqrt{69}\sqrt{41}} \Rightarrow \theta = 0.445\text{rad} = 25.5^\circ$

(c) $\cos\theta = \frac{-7}{\sqrt{14}\sqrt{41}} \Longrightarrow \theta = 1.876\text{rad} = 107^\circ$.

6. (a) OUT (b) IN
(c) OUT (d) result is zero-vector, that is, $\mathbf{0}$ (no direction)

Problems for Lecture 4: Vectors I – Answers

1. (a) $\mathbf{A}\times\mathbf{B} = -2\mathbf{i} - 29\mathbf{j} - 11\mathbf{k}$ (b) $\mathbf{C}\times\mathbf{B} = 10\mathbf{i} + 19\mathbf{j} - 8\mathbf{k}$

   (c) $\mathbf{A}\times\mathbf{C} = 5\mathbf{i} - 22\mathbf{j} - 4\mathbf{k}$

2. (i) The scalar product $\mathbf{a}\bullet\mathbf{b} = 2\cdot(-1) + (-3)\cdot 2 + 1\cdot(-4) = -12$ and the magnitudes $|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$ and $|\mathbf{b}| = \sqrt{(-1)^2 + 2^2 + (-4)^2} = \sqrt{21}$ so we find that $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-12}{\sqrt{14}\sqrt{21}}$ implying that $\theta = 2.35$ rad $= 134.4^\circ$ since $(0 \le \theta \le \pi)$.

   (ii) We calculate the vector product using the 3 by 3 vector-product determinant

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -1 & 2 & -4 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}\mathbf{k}$$

$$= (12-2)\mathbf{i} - (-8+1)\mathbf{j} + (4-3)\mathbf{k}$$

$$= 10\mathbf{i} + 7\mathbf{j} + \mathbf{k}.$$

   A self-consistent check is to note that $\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b}) = 0$ and $\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b}) = 0$ as they must be since $\mathbf{a} \perp (\mathbf{a}\times\mathbf{b})$ and $\mathbf{b} \perp (\mathbf{a}\times\mathbf{b})$. The magnitude of the vector product is given by $|\mathbf{a}\times\mathbf{b}| = \sqrt{10^2 + 7^2 + 1^2} = \sqrt{150}$.

   (iii) The magnitude of the vector product $|\mathbf{a}\times\mathbf{b}|$ equals the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$.

   (iv) We find that $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = (10,7,1)\bullet(1,-2,\alpha) = 10\cdot 1 + 7\cdot(-2) + 1\cdot\alpha = \alpha - 4.$ Therefore, when $\alpha = 2$ we have $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = -2.$

   (v) By inspection we see that $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = 0 \Leftrightarrow 4 - \alpha = 0 \Leftrightarrow \alpha = 4.$ From the scalar product being zero we conclude that $\mathbf{c} \perp (\mathbf{a}\times\mathbf{b})$. Therefore, $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$ (remember that $\mathbf{a} \perp (\mathbf{a}\times\mathbf{b})$ and $\mathbf{b} \perp (\mathbf{a}\times\mathbf{b})$). We say that the three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar.

Problems for Lecture 5: Answers

1. Since the line passes through the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, the vector $\mathbf{r}_2 - \mathbf{r}_1 = (8-3)\mathbf{i} + (-5-4)\mathbf{j} = 5\mathbf{i} - 9\mathbf{j}$ specifies the direction for the line. Hence, a parametric vector equation for this line is $\mathbf{r} = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1), \lambda \in \mathbb{R}$ that is, $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j}) + \lambda(5\mathbf{i} - 9\mathbf{j})$ or, if you prefer, $\mathbf{r} = (3+5\lambda)\mathbf{i} + (4-9\lambda)\mathbf{j}$. Call this line A for future reference.

   Written in terms of components, this leads to $x = 3+5\lambda$ and $y = 4-9\lambda$. Isolating λ in these two equations, we find $\lambda = \frac{x-3}{5} = \frac{y-4}{-9}$. Note that the denominators are the associated coordinates of the vector specifying the direction of the lines, that is, the direction ratios for line A.

   Note that the vector $\mathbf{r}_1$ on the right hand side of the original vector equation could be the position vector of <u>*any*</u> point on the line, and that the vector multiplied by λ could be <u>*any*</u> vector parallel to the line, that is, any multiple $\mu(\mathbf{r}_2 - \mathbf{r}_1), \mu \neq 0$ would serve as the direction.

2. The gradient is 3, so the simplest choice of a vector specifying the direction of the line is $\mathbf{d} = (1,3)$. The line passes through the point $\mathbf{r}_0 = (x_0, y_0) = (0,-2)$ so a possible parametric vector equation for this line is $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d} = \lambda\mathbf{i} + (-2+3\lambda)\mathbf{j}, \lambda \in \mathbb{R}$. On coordinate form, this yields $x = \lambda$ and $y = -2+3\lambda$. Hence we find $y = -2+3x$. Call this line B for future reference.

3. (a) The direction rations of line A are $(5,-9)$ and those of line B are $(1,3)$.

   (b) The direction cosines are found by normalising the vector specifying the direction of the line. Hence, since $|\mathbf{r}_2 - \mathbf{r}_1| = \sqrt{5^2 + (-9)^2} = \sqrt{106}$, the direction cosines of line A are $\left(\frac{5}{\sqrt{106}}, \frac{-9}{\sqrt{106}}\right)$. Similarly, since $|\mathbf{d}| = \sqrt{1^2 + 3^2} = \sqrt{10}$, the direction cosines of line B are $\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$.

   (c) Note that if $\mathbf{a} = (a_x, a_y)$ is a vector in $\mathbb{R}^2$ then the vectors $\mathbf{a}_\perp = (-a_y, a_x)$ and $-\mathbf{a}_\perp = (a_y, -a_x)$ are perpendicular to $\mathbf{a}$ since $\mathbf{a}\bullet\mathbf{a}_\perp = a_x \cdot (-a_y) + a_y \cdot a_x = 0$ and $\mathbf{a}\bullet(-\mathbf{a}_\perp) = a_x \cdot a_y + a_y \cdot (-a_x) = 0$. This leads to the following unit normal vectors for lines A and B $\hat{\mathbf{n}}_A = \pm\frac{9}{\sqrt{106}}\mathbf{i} \pm \frac{5}{\sqrt{106}}\mathbf{j}$ and $\hat{\mathbf{n}}_B = \mp\frac{3}{\sqrt{10}}\mathbf{i} \pm \frac{1}{\sqrt{10}}\mathbf{j}$, respectively.

(d) & (e) The angle θ between the lines A and B is the same as the angle between the normals $\hat{\mathbf{n}}_A$ and $\hat{\mathbf{n}}_B$. Since, in general, $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, we find $\cos\theta = \hat{\mathbf{n}}_A \bullet \hat{\mathbf{n}}_B = \dfrac{-22}{\sqrt{1060}}$ yielding $\theta = 2.31\,\text{rad} = 132.5°$. However, since the angle between two lines by definition is the smaller of the two, the answer is $\theta = 180^\circ - 132.5^\circ = 47.5^\circ$.

(f) The perpendicular distance to line A is $p_A = |\hat{\mathbf{n}}_A . \mathbf{r}_A|$, where $\mathbf{r}_A$ is any position vector on line A and similarly, the perpendicular distance to line B from the origin is $p_B = |\hat{\mathbf{n}}_B . \mathbf{r}_B|$, where $\mathbf{r}_B$ is any position vector on line B. Hence, we find that $p_A = \dfrac{47}{\sqrt{106}}$ and that $p_B = \dfrac{2}{\sqrt{10}}$.

4. The direction vectors of the two lines $(2,5,3)$ and $(4,10,6)$ are proportional, and hence, the directions are therefore the same. The second line clearly goes through (10, 22, 13) and direct substitution shows that the first line does too. Since therefore the lines have a point in common, and are in the same direction, they are identical.

Problems for Lecture 6: Answers

1. (a) A normal vector to the plane is given by the coefficients of the three unknowns, that is, $\mathbf{n}_1 = 5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. To determine a unit normal vector, we need to divide by the magnitude of $\mathbf{n}_1$, $|\mathbf{n}_1| = \sqrt{5^2 + (-4)^2 + (-3)^2} = \sqrt{50}$, so the unit normal vector to the plane $\hat{\mathbf{n}}_1 = \dfrac{5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}}{\sqrt{50}} = \dfrac{5}{\sqrt{50}}\mathbf{i} - \dfrac{4}{\sqrt{50}}\mathbf{j} - \dfrac{3}{\sqrt{50}}\mathbf{k}$.
(b) Dividing the equation for the plane by the magnitude of the normal vector yields $\dfrac{5x - 4y - 3z}{\sqrt{50}} = \dfrac{10}{\sqrt{50}} = \sqrt{2}$. In this form, the right-hand-side is the minimal distance from the origin to the plane, that is, $d_O = \sqrt{2}$, see Fact Sheet 4 or Fact Sheet 10.

(c) Choose any point A on the plane, say $\overrightarrow{OA} = (2,0,0)$, found by inserting $y = z = 0$ into the equation for the plane and solving the resulting equation $5x = 10$. The vector from A to P is $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-1,3,5) = -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$. Hence, the minimal distance from the point P to the plane $d_P = \left|\overrightarrow{AP} \bullet \hat{\mathbf{n}}\right| = \left|\dfrac{(-1)\cdot 5 + 3\cdot(-4) + 5\cdot(-3)}{\sqrt{50}}\right| = \dfrac{32}{\sqrt{50}} \approx 4.53$.

2. The line joining the two points has direction $\mathbf{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (7,4,3) = 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and it follows that a unit vector in the direction of the line is $\hat{\mathbf{d}} = \dfrac{\mathbf{d}}{|\mathbf{d}|} = \dfrac{7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}}{\sqrt{74}}$. The vector from, say, A to P is $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. The minimal distance from the point P to the plane is

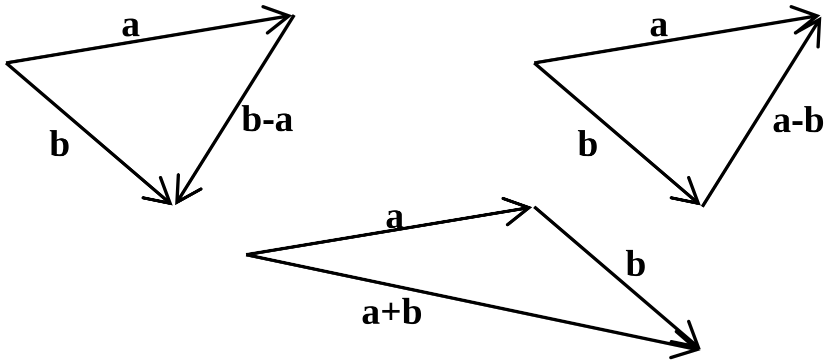
$$d = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right| = \frac{1}{\sqrt{74}}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & -2 \\ 7 & 4 & 3 \end{matrix}\right\| = \left|\frac{-\mathbf{i} - 23\mathbf{j} + 33\mathbf{k}}{\sqrt{74}}\right| = \sqrt{\frac{1619}{74}} \approx 4.68.$$

3. We find a vector $\overrightarrow{A_1A_2}$ joining arbitrary points A_1 from line 1 and A_2. Using $\overrightarrow{OA_1} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OA_2} = \alpha\mathbf{i} + \mathbf{j} + \mathbf{k}$, we find $\overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1} = (\alpha - 1)\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The two lines will intersect when $\overrightarrow{A_1A_2}$ and the two direction vectors $\mathbf{d}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{d}_2 = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ are coplanar, that is, when

$$\overrightarrow{A_1A_2} \bullet (\mathbf{d}_1 \times \mathbf{d}_2) = \det\left(\overrightarrow{A_1A_2}, \mathbf{d}_1, \mathbf{d}_2\right) = \begin{vmatrix} \alpha - 1 & 3 & 2 \\ -1 & 2 & -3 \\ -2 & 1 & -4 \end{vmatrix} = \begin{vmatrix} \alpha + 5 & 0 & 14 \\ 3 & 0 & 5 \\ -2 & 1 & -4 \end{vmatrix} = -1 \cdot \begin{vmatrix} \alpha + 5 & 14 \\ 3 & 5 \end{vmatrix} = 0,$$

that is, when $5\alpha + 25 - 42 = 0 \Leftrightarrow \alpha = 17/5$.

5. (i)



(ii) Since **a** and **b** are non-zero non-parallel vector, any point **c** in the plane can be reached by choosing appropriate numbers $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$. We say that **c** is written as a linear combination of **a** and **b**.

(iii) We have to find a solution to the equations $\begin{matrix} 3 = 1\cdot\lambda + (-1)\cdot\mu = \lambda - \mu \\ 7 = 1\cdot\lambda + 2\cdot\mu = \lambda + 2\mu \end{matrix}$. Subtracting Eq.(1) from Eq. (2) yields $4 = 3\mu \Leftrightarrow \mu = \frac{4}{3}$ and substituting that into Eq. (1) yields $\lambda = 3 + \mu = \frac{13}{3}$, so that $\mathbf{c} = \frac{13}{3}\mathbf{a} + \frac{4}{3}\mathbf{b}$.

Problems for Lecture 7-8: Answers

1. (a) $\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} = 4\cdot 5 - 1\cdot 2 = 18$ and (b) $\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 4\cdot 5 - 2\cdot 1 = 18$ too. The matrix association with case (b) is the transpose of that in case (a). Indeed, if $\mathbf{A}$ is an $n \times n$ matrix, then $\det \mathbf{A} = \det \mathbf{A}^t$ where the matrix $\mathbf{A}^t$ is the transpose of matrix $\mathbf{A}$, that is, the matrix obtained by reflecting the matrix $\mathbf{A}$ across its main diagonal. In other words, the ijth entry in $\mathbf{A}^t$ is equal to the jith entry in $\mathbf{A}$, that is, $a_{ij}^t = a_{ji}$. Hence, the value of a determinant is unchanged if rows and columns are interchanged.

(c) $\begin{vmatrix} 1 & 5 \\ 4 & 2 \end{vmatrix} = 1\cdot 2 - 4\cdot 5 = -18$. The two rows of the determinant in (a) have been reversed. Indeed, the sign of a determinant is reversed if two rows (or two columns) are interchanged.

(d) $\begin{vmatrix} 8 & 2 \\ 2 & 5 \end{vmatrix} = 8\cdot 5 - 2\cdot 2 = 36$. Column 1 of determinant (a) has been multiplied by a factor of 2. Indeed, if the matrix $\mathbf{B}$ is obtained from the matrix $\mathbf{A}$ by multiplying some column (or row) by a number r, $\det \mathbf{B} = r \det \mathbf{A}$.

(e) $\begin{vmatrix} 4 & 2 \\ 5 & 7 \end{vmatrix} = 4\cdot 7 - 5\cdot 2 = 18$. Row 1 of determinant (a) has been added to row 2. Indeed, if the matrix $\mathbf{B}$ is obtained from the matrix $\mathbf{A}$ by adding a numerical multiple of one row (column) to another, $\det \mathbf{B} = \det \mathbf{A}$.

2. (a) We evaluate the determinant of the matrix of coefficients:

$$\det \mathbf{A} = \begin{vmatrix} 8 & 1 & 8 \\ 6 & 4 & 4 \\ 5 & -1 & 6 \end{vmatrix} = 8\begin{vmatrix} 4 & 4 \\ -1 & 6 \end{vmatrix} - 1\begin{vmatrix} 6 & 4 \\ 5 & 6 \end{vmatrix} + 8\begin{vmatrix} 6 & 4 \\ 5 & -1 \end{vmatrix} = 8\cdot 28 - 16 + 8\cdot(-26) = 0.$$ The

determinant of the matrix of coefficients is zero and there is no unique solution. To determine whether there are no solutions or infinitely many solutions, we apply Gauss elimination: multiplying the second equation with -2 and adding to the first equation, and multiplying the second equation with $-3/2$ and adding to the third equation yields the equivalent system:

$$\begin{aligned} -4x_1 \quad - 7x_2 &= -4 \\ 6x_1 + 4x_2 + 4x_3 &= 8 \\ -4x_1 \quad - 7x_2 &= 3 \end{aligned}$$

where the first and third equations are clearly incompatible.

Hence there are no solutions to this system of equations.

(b) The is no unique solution. (If you insist on applying Cramer's rule to arrive at this conclusion, add a third equation $0x_1 + 0x_2 + 0x_3 = 0$ and show that the determinant of the associated matrix of coefficients is zero.) The two equations each specify a plane in $\mathbb{R}^3$. These planes are not parallel (since their respective normal vectors $(4,7,-2)$ and $(1,-3,2)$ are not parallel) so they will intersect in a line. Applying Gauss elimination, multiplying the second equation with -4 and adding to the first equation, we find

$\begin{aligned} 19x_2 - 10x_3 &= -14 \\ x_1 - 3x_2 + 2x_3 &= 6 \end{aligned}$. Now, multiplying the first equation by $3/19$ and adding to the second, we find $\begin{aligned} 19x_2 - 10x_3 &= -14 \\ x_1 \qquad + \frac{8}{19}x_3 &= \frac{72}{19} \end{aligned}$, that is, $\begin{aligned} x_1 &= \frac{72}{19} - \frac{8}{19}x_3 \\ x_2 &= -\frac{14}{19} + \frac{10}{19}x_3 \end{aligned}$ obtained by interchanging the two equations and multiplying the new second equation with $1/19$ and rearranging. Hence, given x_3, x_1 and x_2 are determined by these two equations. We may write this system of equations on vector form:

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 72/19 \\ -14/19 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -8/19 \\ 10/19 \\ 1 \end{pmatrix} \Leftrightarrow \mathbf{r} = \mathbf{r}_0 + \lambda \mathbf{d}$ where λ is a real number, revealing that the solution is a line through the point $\mathbf{r}_0$ along the direction of $\mathbf{d}$.

(c) We evaluate the determinant of the matrix of coefficients:

$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 4 & -4 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ -4 & -4 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 4 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 4 & -4 \end{vmatrix} = 16 + 16 + 0 = 32$. Since $\det \mathbf{A} \neq 0$, there is a unique solution. Applying Cramer's rule, we find

$x_1 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & -4 & -4 \end{vmatrix}}{\det \mathbf{A}} = \dfrac{16 - 2 - 2}{32} = \dfrac{3}{8}$, $x_2 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & -1 & -4 \end{vmatrix}}{\det \mathbf{A}} = \dfrac{2 + 16 - 2}{32} = \dfrac{1}{2}$, and

$x_3 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 4 & -4 & -1 \end{vmatrix}}{\det \mathbf{A}} = \dfrac{2 + 2 - 0}{32} = \dfrac{1}{8}$ so the unique solution is $(x_1, x_2, x_3) = (3/8, 1/2, 1/8)$ which is easily checked by substituting into the original system of equations.

Problems for Lecture 9: Answers

1. (a) Since we may expand the determinant by any row or any column, we choose to expand by the third column to take advantage of the zeros. We find

$$\begin{vmatrix} 4 & 1 & 2 \\ 7 & 2 & 0 \\ -2 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} 7 & 2 \\ -2 & 3 \end{vmatrix} = 2(21+4) = 50.$$

(b) We use property 6 for determinants and multiply the second column by (-2) and add to the third column and then expand by the third column:

$$\begin{vmatrix} 3 & 2 & 4 \\ 5 & 4 & 8 \\ 8 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 0 \\ 5 & 4 & 0 \\ 8 & 2 & 5 \end{vmatrix} = 5\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 5(12-10) = 10.$$

(c) By expanding the determinant by the first column and similarly, the determinant of its minor by the first column and so on, we find

$$\begin{vmatrix} 2 & 15 & -37 & 8 & 11 \\ 0 & 1 & 6 & 23 & -32 \\ 0 & 0 & 4 & 12 & -29 \\ 0 & 0 & 0 & 10 & 20 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 6 & 23 & -32 \\ 0 & 4 & 12 & -29 \\ 0 & 0 & 10 & 20 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 2\cdot 1\begin{vmatrix} 4 & 12 & -29 \\ 0 & 10 & 20 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 2\cdot 1\cdot 4\begin{vmatrix} 10 & 20 \\ 0 & 3 \end{vmatrix} = 2\cdot 1\cdot 4\cdot 10\cdot 3 = 240$$

that is, the product of the diagonal elements. Indeed, in a so-called *triangular matrix*, where the elements below or above the main diagonal are zero, the determinant is simply the product of the (main) diagonal elements. Notice that you may apply property 6 of the determinants and use the last row to create zeros in the last column above the last row, then use the second to last row to create zeros in the second to last column above that row etc. until you finally have the determinant of a *diagonal matrix*, where the only non-zero elements are in the main diagonal. Clearly the determinant of a diagonal matrix is the product of the elements in the main diagonal.

2. (a) Column 3 is 3 times column 1, so the determinant is zero by property 5.

(b) Expanding by the first column yields $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} = -6$.

(c) Row 1 is the sum of rows 2 and 3. Subtracting rows 2 and 3 in succession from row 1 makes all elements in row 1 equal to zero. Hence the determinant is zero.

(d) Expanding by the first row yields $\begin{vmatrix} 0 & 7 & 0 \\ 3 & -5 & 6 \\ 2 & 3 & -4 \end{vmatrix} = -7\begin{vmatrix} 3 & 6 \\ 2 & -4 \end{vmatrix} = -7(-12-12) = 168$.

3. (a) If you change the sign of all elements in the matrix, the sign of the determinant should be reversed because each term has three factors (an odd number). This can also be obtained by applying property 2: $\det(-\mathbf{A}) = (-1)^n \det \mathbf{A}$. For this particular matrix in question 3(a), changing the sign leads to the transpose of the original matrix, that is, $\mathbf{A}^t = -\mathbf{A}$. However, by property 7, $\det \mathbf{A} = \det \mathbf{A}^t$, yielding $\det \mathbf{A} = \det(-\mathbf{A}) = -\det \mathbf{A}$ Hence, the determinant of $\mathbf{A}$ can only be zero.

(b) Using property 8: $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}.$

Let us investigate the second determinant. We assume that $a,b,c,d \neq 0$. Then we find

$$\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

where step 1 involves multiplying rows 1-4 by a,b,c,d respectively (using property 2), step 2 involves applying property 2 on column 4 and step 3 involves an odd number of column exchanges, which reverses the sign (property 1). Substituting this result into the equation above, we see that the determinant of the original matrix is zero! (There are probably other ways (may be even simpler) of obtaining this result.)

If, say $a = 0$, is the determinant zero? Let me know as I haven't done the calculation myself! ☺

4. (a) $3\cdot 2 = 6.$ (b) $3-1 = 2.$ (c) $3\cdot 2\cdot(3-1) = 12$.

(d) $4\cdot(3\cdot 2) = 24$, $4-1 = 3.$, $4\cdot 3\cdot 2\cdot(4-1) = 72.$

(e) The determinant of an $n\times n$ matrix is defined as the sum of n terms, where each term contain the determinant of an $(n-1)\times(n-1)$ matrix. Such a determinant is defined as the sum of $(n-1)$ terms, where each term contains the determinant of an $(n-2)\times(n-2)$ matrix and so on. The total number of terms in the sum: $n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 2\cdot 1 = n!$ The number of multiplications in each term is $(n-1)$. Hence to total number of multiplications that has to be performed is $n!(n-1)$.

(f) $n!(n-1) = 25!24 = 3.72\times 10^{26}$.

(g) Total number of operations is the number of multiplications added with the number of additions, in total $n!(n-1) + n! - 1 = n!n - 1 = 3.88\cdot 10^{26}$. The time required to evaluate the determinant $3.88\cdot 10^{26}/3\cdot 10^{15}\text{s} \approx 1.3\cdot 10^{11}\text{s} \approx 4{,}101$ years

Problems for Lecture 10: Answers

Consider the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A}$ is an $n \times n$ matrix of coefficients.

The homogeneous equation always has the trivial solution, $\mathbf{x} = \mathbf{0}$.

If $\det \mathbf{A} \neq 0$, the homogeneous equation has *only the trivial solution,* $\mathbf{x} = \mathbf{0}$.

If $\det \mathbf{A} = 0$, the homogeneous equation has *additional non-trivial solutions,* $\mathbf{x} \neq \mathbf{0}$.

1. (a) The determinant of the matrix of coefficients $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 12 - 10 = 2 \neq 0$. Therefore, $x = y = 0$ is the only solution to the homogeneous eq. It has no non-trivial solutions.

(b) The determinant of the matrix of coefficients $\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix} = 6 + 35 = 41 \neq 0$. Therefore, the homogeneous equation has no non-trivial solutions. The only solution is $x = y = 0$.

(c) The determinant of the matrix of coefficients $\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 12 - 12 = 0$. The two equations are proportional: Eq.(1) is $1.5\times$ Eq.(2). Hence, one of the equations is redundant, say Eq.(2). Multiplying Eq.(1) by $\frac{1}{3}$ yields $2x + y = 0$, which defines the line of solutions (non-trivial as well as the trivial solution). On parametric vector form, the eq. for the line of solutions is $\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$. Note that $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(d) The determinant of the matrix of coefficients $\begin{vmatrix} 1.4 & -1.2 \\ -2.1 & 1.8 \end{vmatrix} = 2.52 - 2.52 = 0$. The two equations are proportional: Eq.(2) is $-1.5\times$Eq.(1) and therefore one of the equations is redundant, say Eq.(1). Multiplying Eq.(2) by $\frac{1}{1.8}$ yields $-\frac{7}{6}x + y = 0$ which defines the line of solutions. On parametric vector form, the equation for the line of solutions is $\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \frac{7}{6} \end{pmatrix}, \lambda \in \mathbb{R}$. Note that $\begin{pmatrix} 1.4 & -1.2 \\ -2.1 & 1.8 \end{pmatrix}\begin{pmatrix} 1 \\ 7/6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2. (a) The determinant of the matrix of coefficients $\begin{vmatrix} 8 & 1 & 8 \\ 6 & 4 & 4 \\ 5 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 6 & 4 & 4 \\ 5 & -1 & 6 \end{vmatrix} = 0$ where we have added $(-1)\times$row 3 to row 1 and $-\frac{1}{2}\times$row 2 to row 1. Hence, the system has non-trivial solutions in addition to the trivial solution.

(b) The determinant of the matrix of coefficients is easily found by adding $2\times$row 2 to row 1 and expanding by column 2: $\begin{vmatrix}5 & 2 & 2\\ 1 & -1 & 4\\ 7 & 0 & 1\end{vmatrix} = \begin{vmatrix}7 & 0 & 10\\ 1 & -1 & 4\\ 7 & 0 & 1\end{vmatrix} = -1\cdot\begin{vmatrix}7 & 10\\ 7 & 1\end{vmatrix} = 63 \neq 0$.

Hence, the homogeneous equation has only the trivial solution $p = q = r = 0$. No non-trivial solutions exist.

(c) The determinant of the matrix of coefficients $\begin{vmatrix}12 & -16 & 2 & 8\\ -6 & 6 & 14 & -3\\ 10 & 10 & -7 & -5\\ 11 & -18 & 2 & 9\end{vmatrix} = 0$ since column 2 and column 4 are proportional (column 2 = $-2\times$ column 4). Hence the homogeneous equation has non-trivial solutions in addition to the trivial solution $\mathbf{x} = \mathbf{0}$.

3. Let us first evaluate the vector products:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -3\\ 7 & -2 & 4\end{vmatrix} = (4-6)\mathbf{i} - (8+21)\mathbf{j} + (-4-7)\mathbf{k} = -2\mathbf{i} - 29\mathbf{j} - 11\mathbf{k},$$

$$\mathbf{c}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 0 & 5\\ 7 & -2 & 4\end{vmatrix} = (0+10)\mathbf{i} - (16-35)\mathbf{j} + (-8-0)\mathbf{k} = 10\mathbf{i} + 19\mathbf{j} - 8\mathbf{k},$$

$$\mathbf{c}\times\mathbf{a} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 0 & 5\\ 2 & 1 & -3\end{vmatrix} = (0-5)\mathbf{i} - (-12-10)\mathbf{j} + (4-0)\mathbf{k} = -5\mathbf{i} + 22\mathbf{j} + 4\mathbf{k},$$

$$\mathbf{b}\times\mathbf{c} = -\mathbf{c}\times\mathbf{b} = -10\mathbf{i} - 19\mathbf{j} + 8\mathbf{k}.$$

Hence, we find that

(a) $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = \left(-2\mathbf{i} - 29\mathbf{j} - 11\mathbf{k}\right)\bullet\left(4\mathbf{i} + 5\mathbf{k}\right) = (-2)\cdot 4 + (-29)\cdot 0 + (-11)\cdot 5 = -63.$

(b) $\mathbf{a}\bullet\left(\mathbf{c}\times\mathbf{b}\right) = \left(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\right)\bullet\left(10\mathbf{i} + 19\mathbf{j} - 8\mathbf{k}\right) = 2\cdot 10 + 1\cdot 19 + (-3)\cdot(-8) = 63.$

(c) $\mathbf{b}\bullet\left(\mathbf{c}\times\mathbf{a}\right) = \left(7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\right)\bullet\left(-5\mathbf{i} + 22\mathbf{j} + 4\mathbf{k}\right) = 7\cdot(-5) + (-2)\cdot 22 + 4\cdot 4 = -63.$

Note that the absolute values are all the same, but the sign changes when cyclic order is not maintained.

(d) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & -29 & -11\\ 4 & 0 & 5\end{vmatrix}=-145\mathbf{i}-(-10+44)\mathbf{j}+116\mathbf{k}=-145\mathbf{i}-34\mathbf{j}+116\mathbf{k}$,

(e) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -3\\ -10 & -19 & 8\end{vmatrix}=(8-57)\mathbf{i}-(16-30)\mathbf{j}+(-38+10)\mathbf{k}=-49\mathbf{i}+14\mathbf{j}-28\mathbf{k}$.

4. (a) Let us consider the equation $c_1\mathbf{a}+c_2\mathbf{b}+c_3\mathbf{c}=\mathbf{0}$. On component form we have

$$\begin{aligned}2c_1+7c_2+4c_3&=0\\ c_1-2c_2&=0\\ -3c_1+4c_2+5c_3&=0\end{aligned}$$

. However, the determinant of the matrix of coefficient, whose columns are $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$, respectively, $\det(\mathbf{a},\mathbf{b},\mathbf{c})=\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})=-63\neq 0$ so the trivial solution $c_1=c_2=c_3=0$ is the only solution. Hence, the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are linearly independent.

(b) *Linear dependence* implies that the three vectors are in the same plane; they are coplanar. *Linear independence* implies that the three vectors are not coplanar. The determinant formed from the components of the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ is the triple scalar product, that is, $\det(\mathbf{a},\mathbf{b},\mathbf{c})=\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$ and represents the volume of the parallelepiped whose sides are given by the three vectors. The volume is zero when the vectors are coplanar, that is, when the vectors are linearly dependent. The volume is non-zero when the vectors are linearly independent.

(c) The determinant with columns $\mathbf{a},\mathbf{b},\mathbf{c}$, expanded by the second row is

$$\det(\mathbf{a},\mathbf{b},\mathbf{c})=\begin{vmatrix}2+\alpha & 7 & 4\\ 1 & -2 & 0\\ -3 & 4 & 5\end{vmatrix}=-1\cdot\begin{vmatrix}7 & 4\\ 4 & 5\end{vmatrix}-2\cdot\begin{vmatrix}2+\alpha & 4\\ -3 & 5\end{vmatrix}=-63-10\alpha$$. Therefore,

$\det(\mathbf{a},\mathbf{b},\mathbf{c})=0\Leftrightarrow\alpha=-6.3$, making the vectors linearly dependent.

5. We will use the **b**ac-**c**ab rule $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\bullet\mathbf{c})-\mathbf{c}(\mathbf{a}\bullet\mathbf{b})$.

Hence, by applying this rule on the three triple vector products, we find

$$\begin{aligned}&\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\\ &\mathbf{b}(\mathbf{a}\bullet\mathbf{c})-\mathbf{c}(\mathbf{a}\bullet\mathbf{b})+\mathbf{c}(\mathbf{b}\bullet\mathbf{a})-\mathbf{a}(\mathbf{b}\bullet\mathbf{c})+\mathbf{a}(\mathbf{c}\bullet\mathbf{b})-\mathbf{b}(\mathbf{c}\bullet\mathbf{a})=0.\end{aligned}$$

Problems for Lecture 11: Answers

1. $(a)\quad 3\mathbf{A} = 3\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 12 & 18 \end{pmatrix},$

$(b)\quad \mathbf{A}+\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 9 & 10 \end{pmatrix},$

$(c)\quad 3\mathbf{B}-2\mathbf{C} = 3\begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix} - 2\begin{pmatrix} -4 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 15 & 12 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 14 & -7 \\ 15 & 10 \end{pmatrix}.$

2. (a) $\mathbf{r}_1 = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + 3\cdot 4 \\ 4\cdot 3 + 6\cdot 4 \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \end{pmatrix},$

(b) $\mathbf{r}_2 = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + (-1)\cdot 4 \\ 5\cdot 3 + 4\cdot 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 31 \end{pmatrix},$

(c) $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 = \begin{pmatrix} 18 \\ 36 \end{pmatrix} + \begin{pmatrix} 2 \\ 31 \end{pmatrix} = \begin{pmatrix} 20 \\ 67 \end{pmatrix},$

(d) $\mathbf{r}_4 = \begin{pmatrix} 4 & 2 \\ 9 & 10 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4\cdot 3 + 2\cdot 4 \\ 9\cdot 3 + 10\cdot 4 \end{pmatrix} = \begin{pmatrix} 20 \\ 67 \end{pmatrix}.$

The transformation effected by the sum of the two matrices $(\mathbf{A}+\mathbf{B})$ is the same as the sum of the two transformations $\mathbf{A}$ and $\mathbf{B}$.

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then $\mathbf{Ar} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$ and

$\mathbf{Br} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{pmatrix}$ so that

$\mathbf{Ar}+\mathbf{Br} = \begin{pmatrix} a_{11}x + a_{12}y + b_{11}x + b_{12}y \\ a_{21}x + a_{22}y + b_{21}x + b_{22}y \end{pmatrix}$, using the property of matrix addition

$= \begin{pmatrix} (a_{11}+b_{11})x + (a_{12}+b_{12})y \\ (a_{21}+b_{21})x + (a_{22}+b_{22})y \end{pmatrix}$, using the property for real numbers

$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, using then definition of matrix multiplication

$= \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix}$, using the definition of matrix addition

$= (\mathbf{A}+\mathbf{B})\mathbf{r}.$

3. (a) $\mathbf{AB}=\begin{pmatrix}2&3\\4&6\end{pmatrix}\begin{pmatrix}2&-1\\5&4\end{pmatrix}=\begin{pmatrix}2\cdot2+3\cdot5&2\cdot(-1)+3\cdot4\\4\cdot2+6\cdot5&4\cdot(-1)+6\cdot4\end{pmatrix}=\begin{pmatrix}19&10\\38&20\end{pmatrix}$,

(b) $\mathbf{BC}=\begin{pmatrix}2&-1\\5&4\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}2\cdot(-4)+(-1)\cdot0&2\cdot2+(-1)\cdot1\\5\cdot(-4)+4\cdot0&5\cdot2+4\cdot1\end{pmatrix}=\begin{pmatrix}-8&3\\-20&14\end{pmatrix}$,

(c) $\mathbf{CB}=\begin{pmatrix}-4&2\\0&1\end{pmatrix}\begin{pmatrix}2&-1\\5&4\end{pmatrix}=\begin{pmatrix}(-4)\cdot2+2\cdot5&(-4)\cdot(-1)+2\cdot4\\0\cdot2+1\cdot5&0\cdot(-1)+1\cdot4\end{pmatrix}=\begin{pmatrix}2&12\\5&4\end{pmatrix}$,

(d) $\mathbf{AC}=\begin{pmatrix}2&3\\4&6\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}2\cdot(-4)+3\cdot0&2\cdot2+3\cdot1\\4\cdot(-4)+6\cdot0&4\cdot2+6\cdot1\end{pmatrix}=\begin{pmatrix}-8&7\\-16&14\end{pmatrix}$,

(e) $(\mathbf{AB})\mathbf{C}=\begin{pmatrix}19&10\\38&20\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}19\cdot(-4)+10\cdot0&19\cdot2+10\cdot1\\38\cdot(-4)+20\cdot0&38\cdot2+20\cdot1\end{pmatrix}=\begin{pmatrix}-76&48\\-152&96\end{pmatrix}$,

(f) $\mathbf{A}(\mathbf{BC})=\begin{pmatrix}2&3\\4&6\end{pmatrix}\begin{pmatrix}-8&3\\-20&14\end{pmatrix}=\begin{pmatrix}2\cdot(-8)+3\cdot(-20)&2\cdot3+3\cdot14\\4\cdot(-8)+6\cdot(-20)&4\cdot3+6\cdot14\end{pmatrix}=\begin{pmatrix}-76&48\\-152&96\end{pmatrix}$,

(g) $(\mathbf{A}+\mathbf{B})\mathbf{C}=\begin{pmatrix}4&2\\9&10\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}4\cdot(-4)+2\cdot0&4\cdot2+2\cdot1\\9\cdot(-4)+10\cdot0&9\cdot2+10\cdot1\end{pmatrix}=\begin{pmatrix}-16&10\\-36&28\end{pmatrix}$,

(h) $\mathbf{AC}+\mathbf{BC}=\begin{pmatrix}-8&7\\-16&14\end{pmatrix}+\begin{pmatrix}-8&3\\-20&14\end{pmatrix}=\begin{pmatrix}-16&10\\-36&28\end{pmatrix}$.

These special cases illustrate general properties of matrix manipulation,

Matrix multiplication is associative $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$ ((e) and (f)).

Matrix multiplication is distributive $(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}$ ((g) and (h))

Matrix multiplication is **not,** in general, commutative $\mathbf{BC}\neq\mathbf{CB}$ ((b) and (c)).

4. Matrix multiplication is only defined between matrices $\mathbf{A}$ and $\mathbf{B}$ if the number of a columns in the matrix $\mathbf{A}$ equals the number of rows in the matrix $\mathbf{B}$. For example, if $\mathbf{A}$ is an $m\times p$ matrix and $\mathbf{B}$ is a $p\times n$ matrix, the matrix product is well-defined and $\mathbf{AB}$ is an $m\times n$ matrix. Note that $\mathbf{BA}$ is **not** well-defined unless $n=m$ in which case $\mathbf{BA}$ is a $p\times p$ matrix. $\mathbf{P}$ is a 2×4 matrix, $\mathbf{Q}$ is a 3×2 matrix, and $\mathbf{R}$ is a 3×3 matrix. Hence only $\mathbf{QP}$ (a 3×4 matrix), $\mathbf{RQ}$ (a 3×3 matrix), and $\mathbf{RR}=\mathbf{R}^2$ (a 3×3 matrix) are well-defined. All other combinations are meaningless. We find:

$$\mathbf{QP}=\begin{pmatrix}2&4\\1&-1\\3&-1\end{pmatrix}\begin{pmatrix}2&3&1&-4\\2&1&0&5\end{pmatrix}=\begin{pmatrix}2\cdot2+4\cdot2&2\cdot3+4\cdot1&2\cdot1+4\cdot0&2\cdot(-4)+4\cdot5\\1\cdot2+(-1)\cdot2&1\cdot3+(-1)\cdot1&1\cdot1+(-1)\cdot0&1\cdot(-4)+(-1)\cdot5\\3\cdot2+(-1)\cdot2&3\cdot3+(-1)\cdot1&3\cdot1+(-1)\cdot0&3\cdot(-4)+(-1)\cdot5\end{pmatrix}$$

$$=\begin{pmatrix}12&10&2&12\\0&2&1&-9\\4&8&3&-17\end{pmatrix}$$

$$\mathbf{RQ}=\begin{pmatrix}2&1&3\\4&-1&-2\\-1&0&1\end{pmatrix}\begin{pmatrix}2&4\\1&-1\\3&-1\end{pmatrix}=\begin{pmatrix}2\cdot2+1\cdot1+3\cdot3&2\cdot4+1\cdot(-1)+3\cdot(-1)\\4\cdot2+(-1)\cdot1+(-2)\cdot3&4\cdot4+(-1)\cdot(-1)+(-2)\cdot(-1)\\(-1)\cdot2+0\cdot1+1\cdot3&(-1)\cdot4+0\cdot(-1)+1\cdot(-1)\end{pmatrix}=\begin{pmatrix}14&4\\1&19\\1&-5\end{pmatrix}$$

$$\mathbf{R}^2=\begin{pmatrix}2&1&3\\4&-1&-2\\-1&0&1\end{pmatrix}\begin{pmatrix}2&1&3\\4&-1&-2\\-1&0&1\end{pmatrix}$$

$$=\begin{pmatrix}2\cdot2+1\cdot4+3\cdot(-1)&2\cdot1+1\cdot(-1)+3\cdot0&2\cdot3+1\cdot(-2)+3\cdot1\\4\cdot2+(-1)\cdot4+(-2)\cdot(-1)&4\cdot1+(-1)\cdot(-1)+(-2)\cdot0&4\cdot3+(-1)\cdot(-2)+(-2)\cdot1\\(-1)\cdot2+0\cdot4+1\cdot(-1)&(-1)\cdot1+0\cdot(-1)+1\cdot0&(-1)\cdot3+0\cdot(-2)+1\cdot1\end{pmatrix}$$

$$=\begin{pmatrix}5&1&7\\6&5&12\\-3&-1&-2\end{pmatrix}.$$

5. Define the inverse matrix $\mathbf{A}^{-1}=\begin{pmatrix}\underset{\sim}{a}_{11}&\underset{\sim}{a}_{12}\\\underset{\sim}{a}_{21}&\underset{\sim}{a}_{22}\end{pmatrix}$. This matrix must satisfy $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$. The latter eq. implies $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}\Leftrightarrow\begin{pmatrix}9&6\\5&3\end{pmatrix}\begin{pmatrix}\underset{\sim}{a}_{11}&\underset{\sim}{a}_{12}\\\underset{\sim}{a}_{21}&\underset{\sim}{a}_{22}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, equivalent of a system of 4 equations with 4 unknowns (actually 2 × 2 Eqs. With 2 unknowns):

$$\begin{matrix}9\underset{\sim}{a}_{11}+6\underset{\sim}{a}_{21}=1 & \underset{\sim}{a}_{12}+6\underset{\sim}{a}_{22}=0\\5\underset{\sim}{a}_{11}+3\underset{\sim}{a}_{21}=0 & 5\underset{\sim}{a}_{12}+3\underset{\sim}{a}_{22}=1\end{matrix}$$ These are solve, for example by using Cramer's rule:

$$\underset{\sim}{a}_{11}=\frac{\begin{vmatrix}1&6\\0&3\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{3}{-3}=-1;\ \underset{\sim}{a}_{21}=\frac{\begin{vmatrix}9&1\\5&0\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{-5}{-3}=\frac{5}{3};\ \underset{\sim}{a}_{12}=\frac{\begin{vmatrix}0&6\\1&3\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{-6}{-3}=2;\ \underset{\sim}{a}_{22}=\frac{\begin{vmatrix}9&0\\5&1\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{9}{-3}=-3.$$

We check that $\begin{pmatrix}-1&2\\\frac{5}{3}&-3\end{pmatrix}\begin{pmatrix}9&6\\5&3\end{pmatrix}=\begin{pmatrix}9&6\\5&3\end{pmatrix}\begin{pmatrix}-1&2\\\frac{5}{3}&-3\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, confirming that indeed the inverse matrix is $\mathbf{A}^{-1}=\begin{pmatrix}-1&2\\\frac{5}{3}&-3\end{pmatrix}$.

6. (a) $3\mathbf{M} = 3\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 9 & 6 \end{pmatrix}$, (b) $\det \mathbf{M} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 4 \cdot 2 - 3 \cdot 1 = 5$,

(c) $\det(3\mathbf{M}) = \begin{vmatrix} 12 & 3 \\ 9 & 6 \end{vmatrix} = 12 \cdot 6 - 9 \cdot 3 = 45 = 3^2 \cdot 5.$

A determinant is multiplied by a factor r if all elements of one row (or column) are multiplied by r (see property 5, p 64 in Notes). Therefore, if all elements of all *n* rows are multiplied by r, which is what happens if the parent matrix is multiplied by a factor r, the determinant will be multiplied by r^n, that is, if $\mathbf{A}$ is an $n \times n$ matrix, then $\det(r\mathbf{A}) = r^n \det \mathbf{A}$.

7. In 2D, a vector can be specified by its Cartesian coordinates $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ or so-called polar coordinates, i.e., magnitude and angle ϕ with positive *x*-axes $\mathbf{r} = \begin{pmatrix} |\mathbf{r}|\cos\phi \\ |\mathbf{r}|\sin\phi \end{pmatrix}$. The magnitude $|\mathbf{r}| = \sqrt{3^2 + 4^2} = 5$ and the angle $\mathbf{r}$ makes with the positive *x*-axis $\tan\phi = \frac{y}{x} = \frac{4}{3}$ so $\phi = \tan^{-1}\frac{4}{3} = 53.13^\circ$. Hence after rotations with, the magnitude invariant but the angle is increased with $\theta = 60^\circ$ and 90°, respectively. We find $\begin{pmatrix} |\mathbf{r}|\cos(\phi+\theta) \\ |\mathbf{r}|\sin(\phi+\theta) \end{pmatrix}$ equal $\begin{pmatrix} 5\cos(113.13^\circ) \\ 5\sin(113.13^\circ) \end{pmatrix} = \begin{pmatrix} -1.96 \\ 4.60 \end{pmatrix}$ and $\begin{pmatrix} 5\cos(143.13^\circ) \\ 5\sin(143.13^\circ) \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

We will later see the by applying the matrix $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ to a vector $\mathbf{r}$ rotates it through an angle of θ counter-clock wise. We note that $\mathbf{R}_{60^\circ} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ and $\mathbf{R}_{90^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence $\mathbf{R}_{60^\circ}\mathbf{r} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 - 2\sqrt{3} \\ 3\sqrt{3}/2 + 2 \end{pmatrix} \approx \begin{pmatrix} -1.964 \\ 4.598 \end{pmatrix}$ and $\mathbf{R}_{90^\circ}\mathbf{r} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$. Note that indeed the matrix $\mathbf{R}_{90^\circ}$ maps a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ into $\begin{pmatrix} -y \\ x \end{pmatrix}$ and they are indeed perpendicular $\begin{pmatrix} x \\ y \end{pmatrix} \bullet \begin{pmatrix} -y \\ x \end{pmatrix} = -xy + xy = 0$. For your own amusement, check that the matrix $\mathbf{R}_{-90^\circ}$ maps $\begin{pmatrix} x \\ y \end{pmatrix}$ into $\begin{pmatrix} y \\ -x \end{pmatrix}$ which are also perpendicular.

Problems to Lecture 12: Answers

1. We use the definition of a matrix products and notice that when $\mathbf{A}$ is an $m \times p$ matrix and $\mathbf{B}$ is an $p \times n$ matrix, then $\mathbf{AB}$ is an $m \times n$ matrix.

(a) $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = 1\cdot 5 + 2\cdot 6 = 17$, (b) $\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = 3\cdot 5 + 4\cdot 6 = 39$,

(c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 6 \\ 3\cdot 5 + 4\cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$, (d) $\begin{pmatrix} 5 \\ 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 5\cdot 1 & 5\cdot 2 \\ 6\cdot 1 & 6\cdot 2 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 6 & 12 \end{pmatrix}$,

(e) $\begin{pmatrix} 1 & -3 & 5 & -7 \end{pmatrix}\begin{pmatrix} 8 \\ 6 \\ 4 \\ 2 \end{pmatrix} = 1\cdot 8 - 3\cdot 6 + 5\cdot 4 - 7\cdot 2 = -4$,

(f) $$\begin{pmatrix} 8 \\ 6 \\ 4 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & -3 & 5 & -7 \end{pmatrix} = \begin{pmatrix} 8\cdot 1 & 8\cdot(-3) & 8\cdot 5 & 8\cdot(-7) \\ 6\cdot 1 & 6\cdot(-3) & 6\cdot 5 & 6\cdot(-7) \\ 4\cdot 1 & 4\cdot(-3) & 4\cdot 5 & 4\cdot(-7) \\ 2\cdot 1 & 2\cdot(-3) & 2\cdot 5 & 2\cdot(-7) \end{pmatrix} = \begin{pmatrix} 8 & -24 & 40 & -56 \\ 6 & -18 & 30 & -42 \\ 4 & -12 & 20 & -28 \\ 2 & -6 & 10 & -14 \end{pmatrix}.$$

2. The 2×2 matrix that represent a rotation in $\mathbb{R}^2$ about the origin by some angle θ, is given by $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where we define the *positive* rotation direction of rotation as *anti-clockwise* (and hence the negative direction of rotation as clockwise).

(a)

$$\mathbf{R}_{+45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix};$$

$$\mathbf{R}_{+90^\circ} = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) $\mathbf{R}^2_{+45^\circ} = \mathbf{R}_{+45^\circ}\mathbf{R}_{+45^\circ} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{R}_{+90^\circ}$.

(*c*) $\mathbf{R}^2_{+90^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{R}_{+180^\circ}$; $\mathbf{R}^2_{-90^\circ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{R}_{-180^\circ}$.

$\mathbf{R}_{+180^\circ} = \mathbf{R}_{-180^\circ}$ correspond to axes inversion: both the *x*- and *y*-axis change direction.

3. (a) Since the rotation is clockwise, the angle is negative. Hence, we find

$$\mathbf{R}_{-|\theta|} = \begin{pmatrix} \cos\left(-\sin^{-1}(4/5)\right) & -\sin\left(-\sin^{-1}(4/5)\right) \\ \sin\left(-\sin^{-1}(4/5)\right) & \cos\left(-\sin^{-1}(4/5)\right) \end{pmatrix} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

and we notice that the numerical value of $\theta = \sin^{-1}(4/5) = 0.927\ \boldsymbol{rad} = 53.13^\circ$.

(b) We find the matrix associated with 45° anti-clockwise rotation: $\mathbf{R}_{+45^\circ} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Therefore, the matrix representing a clockwise rotation of $\theta = \sin^{-1}(4/5)$ followed by an anti-clockwise rotation of 45° is

$$\mathbf{R}_{\mathbf{net}} = \mathbf{R}_{+45^\circ}\mathbf{R}_{-|\theta|} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix}$$

This represents a net *clockwise* rotation of 8.13°, that is, $\mathbf{R}_{-8.13^\circ}$.

(c) We find $\mathbf{R}_{-|\theta|}\mathbf{R}_{+45^\circ} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix}$, which is the same as the result in 3(b). In general, matrix products of rotation matrices are commutative due to the nature of the operation.

(d) The inverse of the matrix $\mathbf{R}_{\text{net}} = \mathbf{R}_{-8.13^\circ}$ must be the reverse rotation (indeed, $\mathbf{R}_{\theta}^{-1} = \mathbf{R}_{-\theta}$), that is, $\mathbf{R}_{\mathbf{net}}^{-1} = \mathbf{R}_{+8.13^\circ} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix}$. To confirm this, note that

$$\mathbf{R}_{\mathbf{net}}\mathbf{R}_{\mathbf{net}}^{-1} = \frac{1}{50}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \&\ \mathbf{R}_{\boldsymbol{net}}^{-1}\mathbf{R}_{\boldsymbol{net}} = \frac{1}{50}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. (a) All elements above the leading diagonal are zero. Therefore, the determinant is the product of the diagonal elements: $\det \mathbf{A} = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 \\ 5 & 4 & 3 & 0 \\ 3 & 2 & 1 & 2 \end{vmatrix} = 4\cdot(-1)\cdot 3\cdot 2 = -24$.

(b) We expand the determinant of the matrix $\mathbf{B}$ by the 4th column:

$$\det \mathbf{B} = 2\begin{vmatrix} 4 & 0 & 1 \\ 6 & -1 & 0 \\ 5 & 4 & 3 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 & 1 \\ 6 & -1 & 0 \\ -7 & 4 & 0 \end{vmatrix} = 2\cdot 1\begin{vmatrix} 6 & -1 \\ -7 & 4 \end{vmatrix} = 2\cdot(24-7) = 34.$$

↑ ↑ ↑ .

Expanded by 4th column. Adding $(-3)\times$ ***row1 to row3. Expanded by 3rd column.***

Problems for Lecture 13: Answers

1. (a) Since $\det \mathbf{A}_1 = \begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix} = 4 \neq 0$, the matrix $\mathbf{A}_1$ is non-singular. We find the inverse

$$\mathbf{A}_1^{-1} = \frac{\text{adj } \mathbf{A}_1}{\det \mathbf{A}_1} = \frac{1}{\det \mathbf{A}_1}\begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}^t = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Please check that $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{A}_1\mathbf{A}_1^{-1} = \mathbf{I}$ ☺.

(b) Since $\det \mathbf{A}_2 = \begin{vmatrix} 6 & -4 \\ -3 & 2 \end{vmatrix} = 12 - (-3)\cdot(-4) = 0$, the matrix $\mathbf{A}_2$ is singular.

(c) Since $\det \mathbf{A}_3 = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1 \neq 0$, the matrix $\mathbf{A}_3$ is non-singular. We find the inverse

$$\mathbf{A}_3 = \frac{\text{adj } \mathbf{A}_3}{\det \mathbf{A}_3} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}^t = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}.$$ Again check that $\mathbf{A}_3^{-1}\mathbf{A}_3 = \mathbf{A}_3\mathbf{A}_3^{-1} = \mathbf{I}$ ☺.

2 (a) The matrix $\mathbf{B}_1$ is singular. Column 2 is $2\times$ column 3, implying $\det \mathbf{B}_1 = 0$.

(b) Expanding by the first row we find $\det \mathbf{B}_2 = -7 \cdot \begin{vmatrix} 3 & 6 \\ 5 & 2 \end{vmatrix} = -7\cdot(-24) = 168 \neq 0$ so the matrix $\mathbf{B}_2$ is non-singular.

(c) The matrix $\mathbf{B}_3$ is singular. Row 3 is ($3\times$ row 1 + $2\times$ row 2), so $\det \mathbf{B}_3 = 0$.

(d) Adding $(-1)\times$ column 2 to column 1 and expanding by column 1 we find

$$\det \mathbf{B}_4 = \begin{vmatrix} 4 & 4 & 4 \\ 2 & 1 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 4 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = -1\cdot \begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} = -8 \neq 0,$$ so $\mathbf{B}_4$ is non-singular.

(e) The matrix $\mathbf{B}_5$ is singular. Rows 1 and 3 are identical, implying $\det \mathbf{B}_5 = 0$.

(f) The matrix $\mathbf{B}_6$ is not square, so it has *no* determinant. Therefore, in principle, the issue of singularity does not arise. However, the term *singular* is sometimes applied to *__any__* matrix that has no inverse. Hence, non-square matrices are singular, since they, by definition, have no inverse.

3. The matrix of the coefficients is $\mathbf{A} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix}$.

(a) Adding $2\times$ row 1 to row 2, $(-1)\times$ row 1 to row 3, and expanding by column 1:

$$\det \mathbf{A} = \begin{vmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 5 & 2 \\ 0 & -4 & -2 \end{vmatrix} = -1\cdot \begin{vmatrix} 5 & 2 \\ -4 & -2 \end{vmatrix} = -1\cdot(-10+8) = 2 \neq 0.$$

(b) The ij th element in the matrix of cofactors is $C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}$. Hence, we find

$$\mathbf{C} = \begin{pmatrix} \begin{vmatrix} 1 & -4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -7 & 2 & -3 \\ -8 & 2 & -4 \\ -11 & 2 & -5 \end{pmatrix}$$

(c) The adjoint matrix is the transpose of the matrix of cofactors. Hence, we have

$$\operatorname{adj} \mathbf{A} = \mathbf{C}^t = \begin{pmatrix} -7 & 2 & -3 \\ -8 & 2 & -4 \\ -11 & 2 & -5 \end{pmatrix}^t = \begin{pmatrix} -7 & -8 & -11 \\ 2 & 2 & 2 \\ -3 & -4 & -5 \end{pmatrix}.$$

(d) The inverse $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix}.$

(e) Since $\det \mathbf{A} \neq 0$, $\mathbf{A}$ is invertible. Therefore, the general solution to the system of linear equation can be found using $\mathbf{Ax} = \mathbf{k} \Leftrightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{k} \Leftrightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{k} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{k}$

Hence $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2}k_1 - 4k_2 - \frac{11}{2}k_3 \\ k_1 + k_2 + k_3 \\ -\frac{3}{2}k_1 - 2k_2 - \frac{5}{2}k_3 \end{pmatrix}.$

Finally, we check (belatedly) that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$:

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} - 8 + \frac{11}{2} & -7 - 4 + 11 & -\frac{21}{2} + 16 - \frac{11}{2} \\ -1 + 2 - 1 & 2 + 1 - 2 & 3 - 4 + 1 \\ \frac{3}{2} - 4 + \frac{5}{2} & -3 - 2 + 5 & -\frac{9}{2} + 8 - \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{AA}^{-1} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2} + 2 - \frac{9}{2} & 4 + 2 - 6 & \frac{11}{2} + 2 - \frac{15}{2} \\ -7 + 1 + 6 & -8 + 1 + 8 & -11 + 1 + 10 \\ \frac{7}{2} - 2 - \frac{3}{2} & 4 - 2 - 2 & \frac{11}{2} - 2 - \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. (a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} - 4 + \frac{11}{2} \\ 1 + 1 - 1 \\ -\frac{3}{2} - 2 + \frac{5}{2} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{35}{2} + 32 \\ 5 - 8 \\ -\frac{15}{2} + 16 \end{pmatrix} = \begin{pmatrix} \frac{29}{2} \\ -3 \\ \frac{17}{2} \end{pmatrix}$ (c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{21}{2} - 16 - \frac{55}{2} \\ 3 + 4 + 5 \\ -\frac{9}{2} - 8 - \frac{25}{2} \end{pmatrix} = \begin{pmatrix} -54 \\ 12 \\ -25 \end{pmatrix}.$

5. Adding $(-5) \times$ row 1 to row 2, adding row 3 to row 1, and expanding by row 2 we find:

$$\det \mathbf{D} = \begin{vmatrix} 0 & 4 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 3 & 2 & -1 \\ 3 & -2 & 5 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 0 & 3 & 0 \\ -1 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} = 2 \cdot (-3) \cdot \begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix} = 6.$$

Problems for Lecture 14: Answers

1. (a) A normal vector to the plane is given by the coefficients of the three unknowns, that is, $\mathbf{n}_1 = 5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. To determine a unit normal vector, we need to divide by the magnitude of $\mathbf{n}_1$, $|\mathbf{n}_1| = \sqrt{5^2 + (-4)^2 + (-3)^2} = \sqrt{50}$, so the unit normal vector to the plane $\hat{\mathbf{n}}_1 = \frac{5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}}{\sqrt{50}} = \frac{5}{\sqrt{50}}\mathbf{i} - \frac{4}{\sqrt{50}}\mathbf{j} - \frac{3}{\sqrt{50}}\mathbf{k}$.

   (b) Dividing the equation for the plane by the magnitude of the normal vector yields $\frac{5x - 4y - 3z}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \sqrt{2}$. In this form, the right-hand-side is the minimal distance from the origin to the plane, that is, $d_O = \sqrt{2}$, see p. 36 in Lecture Notes..

   (c) Choose any point A on the plane, say $\overrightarrow{OA} = (2,0,0)$, found by inserting $y = z = 0$ into the equation for the plane and solving the resulting equation $5x = 10$. The vector from A to P is $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-1,3,5) = -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$. Hence, the minimal distance from the point *P* to the plane $d_P = \left|\overrightarrow{AP} \bullet \hat{\mathbf{n}}\right| = \left|\frac{(-1)\cdot 5 + 3\cdot(-4) + 5\cdot(-3)}{\sqrt{50}}\right| = \frac{32}{\sqrt{50}} \approx 4.53$.

2. By inspection of the equation for the second plane, we see that a normal vector is $\mathbf{n}_2 = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A direction vector **d** for the line of intersection of the two planes is therefore $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -4 & -3 \\ -2 & 1 & 1 \end{vmatrix} = -\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

   A point $\mathbf{r}_0$ that lies on both planes, and therefore on the line of intersection, can be found, for example, by setting $z = 0$ and solving the two resulting equations $5x - 4y = 10$ and $-2x + y = 2$ simultaneously, yielding $x = -6, y = -10$ so that $\mathbf{r}_0 = -6\mathbf{i} - 10\mathbf{j}$. The vector equation of the line of intersection is therefore given by $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d} = (-6\mathbf{i} - 10\mathbf{j}) + \lambda(-\mathbf{i} + \mathbf{j} - 3\mathbf{k})$. Solving the three associated component equiations w.r.t. λ, we find the Cartesian form $\lambda = \frac{x+6}{-1} = \frac{y+10}{1} = \frac{z}{-3}$.

3. The equation for the third plane $x - 2y - z = 14$ is a linear combination of the equations of the other two planes, namely the first plus twice the second. Therefore, in the sense of solving 3 equations with 3 unknowns, the third equation is redundant and the system of linear equations will have the same solutions as before, that is, the line of intersection determined in question 2.

4. The line joining the two points has direction $\mathbf{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (7,4,3) = 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and it follows that a unit vector in the direction of the line is $\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|} = \frac{7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}}{\sqrt{74}}$. The vector from, say, *A* to P is $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. The minimal distance from the point *P* to the plane is

$$d = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right| = \frac{1}{\sqrt{74}} \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & -2 \\ 7 & 4 & 3 \end{matrix} \right\| = \left|\frac{-\mathbf{i} - 23\mathbf{j} + 33\mathbf{k}}{\sqrt{74}}\right| = \sqrt{\frac{1619}{74}} \approx 4.68.$$

5. We find a vector $\overrightarrow{A_1A_2}$ joining arbitrary points A_1 from line 1 and A_2. Using $\overrightarrow{OA_1} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OA_2} = \alpha\mathbf{i} + \mathbf{j} + \mathbf{k}$, we find $\overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1} = (\alpha - 1)\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The two lines will intersect when $\overrightarrow{A_1A_2}$ and the two direction vectors $\mathbf{d}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{d}_2 = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ are coplanar, that is, when

$$\overrightarrow{A_1A_2}\bullet(\mathbf{d}_1 \times \mathbf{d}_2) = \det\left(\overrightarrow{A_1A_2}, \mathbf{d}_1, \mathbf{d}_2\right) = \begin{vmatrix} \alpha-1 & 3 & 2 \\ -1 & 2 & -3 \\ -2 & 1 & -4 \end{vmatrix} = \begin{vmatrix} \alpha+5 & 0 & 14 \\ 3 & 0 & 5 \\ -2 & 1 & -4 \end{vmatrix} = -1\cdot\begin{vmatrix} \alpha+5 & 14 \\ 3 & 5 \end{vmatrix} = 0,$$

that is, when $5\alpha + 25 - 42 = 0 \Leftrightarrow \alpha = 17/5$.

6. (a) In matrix form $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ so the transformation is $\mathbf{T}_a = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

(b) On matrix form $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, that is $x' = 7x - 4y;\ \ y' = 2x.$

7. (a) $(2, \frac{1}{2})$. The associated transformation on matrix form $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

(b) (6, 3). The associated transformation on matrix form $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

(c) $(2, -1)$. The associated transformation on matrix form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

8. (a) $\mathbf{R}^z_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\mathbf{R}^x_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ (c) $\mathbf{R}^y_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$.

9. (a) $\mathbf{R}^z_{+45^\circ}\mathbf{r} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 3 \end{pmatrix}$ (b) $\mathbf{R}^z_{-45^\circ}\mathbf{r} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 3 \end{pmatrix}$.

The magnitude in invariant since $\sqrt{(-\frac{1}{\sqrt{2}})^2 + (\frac{3}{\sqrt{2}})^2 + 3^2} = \sqrt{(\frac{3}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 + 3^2} = \sqrt{14}$.

10. $$\mathbf{R}^x_{-45^\circ}\mathbf{R}^y_{+45^\circ}\mathbf{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 1+\sqrt{2} \\ 1-\sqrt{2} \end{pmatrix}.$$

Problems for Lecture 15: Answers

1. To find the eigenvalues, we solve the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ and to find the associated eigenvectors, we solve the homogeneous equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for each λ.

(i) The characteristic eq.

$$\det(\mathbf{A}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix}=0 \Leftrightarrow (1-\lambda)(2-\lambda)-2\cdot 3=0$$, that is, $\lambda^2-3\lambda-4=0$

$\Leftrightarrow \ \lambda=\dfrac{3\pm\sqrt{9+16}}{2}$ so the eigenvalues are $\lambda_1=4, \lambda_2=-1$. We find

$\lambda_1=4$: $\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1+3y_1=0 \\ 2x_1-2y_1=0 \end{cases} \Leftrightarrow x_1=y_1$ yielding an eigenvector

$\mathbf{x}_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\lambda_2=-1$: $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x_2+3y_2=0 \\ 2x_2+3y_2=0 \end{cases} \Leftrightarrow y_2=-\tfrac{2}{3}x_2$ yielding an eigenvector $\mathbf{x}_2=\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

(ii) We solve

$$\det(\mathbf{B}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0 \Leftrightarrow (1-\lambda)\left((2-\lambda)^2-9\right)=0$$, and thus

$\lambda^3-5\lambda^2-\lambda+5=(\lambda-5)(\lambda^2-1)=0$, so the eigenvalues are $\lambda_1=5, \lambda_2=1, \lambda_3=-1$. We find

$\lambda_1=5$: $\begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1+3y_1=0 \\ 3x_1-3y_1=0 \\ -4z_1=0 \end{cases} \Leftrightarrow \begin{cases} x_1=y_1 \\ z_1=0 \end{cases}$ so $\mathbf{x}_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

$\lambda_2=1$: $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_2+3y_2=0 \\ 3x_2+y_2=0 \\ 0=0 \end{cases} \Leftrightarrow \{x_2=y_2=0$ so $\mathbf{x}_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

$\lambda_3=-1$: $\begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 3x_3+3y_3=0 \\ 3x_3+3y_3=0 \\ 2z_3=0 \end{cases} \Leftrightarrow \begin{cases} y_3=-x_3 \\ z_3=0 \end{cases}$ so $\mathbf{x}_3=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector.

2. First we rotate the <u>coordinate system</u> 45° anti-clockwise by applying the rotation matrix $\mathbf{R}_{-45^\circ} = \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix}$ which is equivalent to rotating a vector clockwise by 45°, hence the negative sign! This transformation is followed by an extension by a factor 2 along the "new" x'-asis by applying the function $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Finally, we rotate the <u>coordinate system</u> 45° clockwise by applying the rotation matrix $\mathbf{R}_{45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$ which is equivalent to rotating a vector anti-clockwise by 45°. The composite transformation is the matrix product of these three matrices:

$$\begin{aligned}
\mathbf{R}_{45^\circ}\mathbf{T}\mathbf{R}_{-45^\circ} &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}.
\end{aligned}$$

3. $\mathbf{Tp} = \mathbf{q} \Leftrightarrow \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_x \\ q_y \end{pmatrix} \Leftrightarrow \begin{pmatrix} t_{11}p_x + t_{12}p_y \\ t_{21}p_x + t_{22}p_y \end{pmatrix} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$. If **p** and **q** are to have the same magnitude, then

$$\begin{aligned}
&p_x^2 + p_y^2 = q_x^2 + q_y^2 \Leftrightarrow \\
&p_x^2 + p_y^2 = (t_{11}p_x + t_{12}p_y)^2 + (t_{21}p_x + t_{22}p_y)^2 \Leftrightarrow \\
&p_x^2 + p_y^2 = (t_{11}^2 + t_{21}^2)p_x^2 + (t_{12}^2 + t_{22}^2)p_y^2 + (t_{11}t_{12} + t_{21}t_{22})2p_xp_y \Leftrightarrow \\
&t_{11}^2 + t_{21}^2 = t_{12}^2 + t_{22}^2 = 1 \text{ and } t_{11}t_{12} + t_{21}t_{22} = 0.
\end{aligned}$$

These conditions imply that the column vectors in **T** are normalised and orthogonal. Hence **T** is an orthogonal matrix. Likewise, these are precisely the same conditions for the transpose of **T** to be its inverse, because

$$\mathbf{T}^t\mathbf{T} = \begin{pmatrix} t_{11} & t_{21} \\ t_{12} & t_{22} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11}^2 + t_{21}^2 & t_{11}t_{12} + t_{21}t_{22} \\ t_{12}t_{11} + t_{22}t_{21} & t_{12}^2 + t_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \mathbf{T}^t = \mathbf{T}^{-1}.$$

4. Since $t_{11}t_{12} + t_{21}t_{22} = 0 \Leftrightarrow t_{22} = -t_{11}t_{12}/t_{21}, t_{21} \neq 0$, we find that $1 = t_{12}^2 + t_{22}^2 = t_{12}^2 + \left(t_{11}t_{12}/t_{21}\right)^2 = t_{12}^2/t_{21}^2\left(t_{21}^2 + t_{11}^2\right) = t_{12}^2/t_{21}^2$, that is $t_{21} = \pm t_{12}$ and $t_{22} = \mp t_{11}$. But 2×2 rotation matrices do not allow for the upper sign option, so the conclusion is that orthogonal matrices represent a *broader* class than rotation matrices. The reason is that orthogonal matrices can include reflections as well as rotations.

5. $\mathbf{T}_1$ is orthogonal since $0.8^2 + 0.6^2 = 1$ and $0.8 \cdot 0.6 + (-0.6 \cdot 0.8) = 0$. It is a rotation matrix with $\theta = -36.87^\circ$. $\mathbf{T}_2$ is orthogonal since $\left(\sqrt{3}/2\right)^2 + \left(1/2\right)^2 = 1$ and $-\sqrt{3}/2 \cdot 1/2 + 1/2 \cdot \sqrt{3}/2 = 0$. However, $\mathbf{T}_2$ is not a a pure rotation matrix. $\mathbf{T}_3$ is not an orthogonal matrix. The column vectors are unit vectors but they are not orthogonal since $1/\sqrt{2} \cdot 1/\sqrt{2} + 1/\sqrt{2} \cdot 1/\sqrt{2} = 1 \neq 0$. Changing the sign on one of the entries in the matrix $\mathbf{T}_3$ would render it orthogonal.

6. $\mathbf{A}_1$ is orthogonal since all the column vectors are normalised and pair-wise orthogonal. $\mathbf{A}_2$ is not orthogonal. The column vectors are normalised but column vector 1 and 3 are not orthogonal. $\mathbf{A}_2$ would, however, be orthogonal if the sign of any one of the four fractional elements were reversed.

7. For an orthogonal matrix **O**, $\mathbf{OO}^t = \mathbf{I}$, where **I** is the identity matrix, so

$$1 = \det \mathbf{I} = \det \mathbf{OO}^t = \det \mathbf{O} \cdot \det \mathbf{O}^t = (\det \mathbf{O})^2$$

where the last step follows because $\det \mathbf{O}^t = \det \mathbf{O}$, see determinant property 6, p. 64 in Lecture Notes.

The conclusion is that $\left(\det \mathbf{O}\right)^2 = 1 \Leftrightarrow \det \mathbf{O} = \pm 1$.