Relativity: Classwork 1

26 November 2009

It is usual to define the shorthand quantities $\beta = \mathrm{v}/c$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, where v is the relative velocity of a moving frame and c is the speed of light. In terms of these quantities we can write the time dilation formula as $t = \gamma t_0$, where t_0 is the *proper time*, the time measured between two events by a clock whose position doesn't change. The length contraction formula is $l = l_0/\gamma$, where l_0 is an object's length measured in a frame in which it is at rest.

1) For what value of β is $\gamma = 2$?

2) Show $\frac{\gamma^2 - 1}{\gamma} = \gamma\beta^2$.

3) Make a plot of γ vs. β for values of β between 0 and 0.9 in steps of 0.1. What happens when $\beta = 1$?

4) Alice is in a spaceship moving at a velocity $\mathrm{v} = \frac{3}{5}c$ with respect to Bob on earth. When she passes earth both of their clocks read $t = 0$.

a) When observers in Bob's frame find Alice has moved 9×10^7m past the earth, what does Alice's clock read. (Use $c = 3\times10^8$m/s.)

b) When Alice's clock reads 0.4s, what does Bob's clock read?

Be careful: remember proper time is recorded by a clock which keeps its position fixed. What are the events? In which frame are the different events recorded in the same place?

Relativity: Classwork 2

3 December 2010

The pole and barn paradox *The traditional paradox:* Suppose a *very* fast runner holding a long pole runs into a barn. In its rest frame the pole is exactly as long as barn in *its* rest frame. In the barn's reference frame, the pole will undergo length contraction, so the farmer will be able to shut the barn door with the pole inside. The runner sees the barn as length contracted, and so feels there is no way the pole can fit in the barn. What happens?

We will analyze a slightly different problem in which the events are clearer. Consider a train and a tunnel, each measured to be length L = 100*m* in their respective rest frame. A rocket is fixed to each end of the train. The driver of the train will fire both rockets when the front of the train just reaches the end of the tunnel.* In the train's frame, the rear rocket easily clears the end of the length contracted tunnel, but in the tunnel frame the train is contracted and the rocket will explode inside the tunnel.

The train and tunnel frames agree that $t = 0$ when the front of the train just enters the tunnel. Take the front of the tunnel to be $x = 0$. The train's speed is $\mathrm{v} = \frac{3}{5}c$.

1) Calculate the length of the tunnel according to the train.

2) In terms of L, v and γ, write down expressions for the space-time coordinates of the following events in the tunnel frame:

  a) The front of the train reaches the exit of the tunnel.
  b) The rear of the train reaches the entrance of the tunnel.

3) Use the Lorentz transformation equations to work out these space-time coordinates of these events in the frame of the train, again in terms of L, v and γ.

4) Calculate the times (in ns) t_f, t'_f, t_b, t'_b for the front and back of the train to reach the ends of the tunnel in each frame.

5) What happens? Is the train entirely in the tunnel or not? Does the rocket explode in the tunnel? (You might need to use the inverse Lorentz transform to find when the train fires its rear rocket in the tunnel frame. It fires at space-time coordinate (t'_f, $-L$) in the train frame.)

Lorentz transformations: $x' = \gamma(x - \mathrm{v}t)$, $t' = \gamma(t - \mathrm{v}x/c^2)$. To find the inverse Lorentz transforms, just swap primes and reverse the sign of v: $x = \gamma(x' + \mathrm{v}t')$, $t = \gamma(t' + \mathrm{v}x'/c^2)$.

* The driver calculates this ahead of time, so we don't need to worry about the time a signal takes to go from the front to the rear of the train.

Relativity: Classwork 3

16 December 2010

1) Alice is arrested for going through a red light. In court she pleads that since she was driving towards the traffic light the blue-shift caused the red light to appear green. The judge accepts the argument, calculates her speed, and changes the charge to speeding. How fast was Alice travelling? [λ_{red} = 630nm, λ_{green} = 530nm, $\lambda\nu = c$]. The Doppler shift formula is

$$\nu_D = \nu_0 \frac{1}{\gamma}\frac{1}{1-\beta}.$$

Hint: first solve the Doppler shift formula for β in terms of the ratio of the wavelengths. Also, prove $\frac{1}{\gamma}\frac{1}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}}$.

2) If the separation between events (x_1, t_1) and (x_2, t_2) is spacelike, find the velocity of the frame in which they occur at the same time. Is this velocity less than c?

3) Express the following quantities in eV, MeV/c^2 etc. ($c = 3\times10^8$m/s, $e = 1.6\times10^{-19}$ C, 1eV = 1.6×10^{-19} J).

(a) Electron mass: $m_e = 9.1 \times 10^{-31}$ kg.

(b) Proton mass: $m_p = 1.67 \times 10^{-27}$ kg.

(c) Total energy of an electron with momentum p = 1MeV/c.

(d) Kinetic energy of a proton with momentum p = 1MeV/c. (Hint: use a binomial expansion.)

(f) Kinetic and total energy of a proton with speed u = 0.8 c. (Hint: first write β in terms of E and p.)

4) How much energy is required to boost a particle with mass *m*

a) from rest to a speed of 0.09c?

b) from 0.9c to 0.99c?

Express your answers in terms of mc^2. How do the answers compare?

5) By what fraction does your rest mass change when you climb 30m to the top of a 10 story building? Does it increase or decrease?

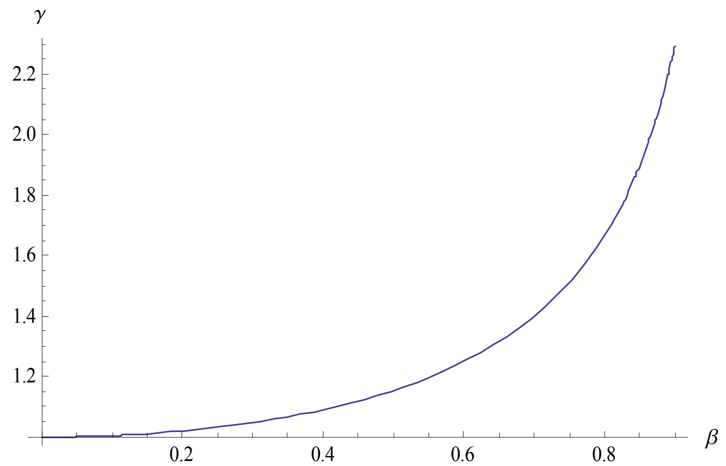
6) If a particle's kinetic energy $(E-mc^2)$ is *n* times its rest energy, find its speed in terms of *n* and *c*.

Relativity: Classwork 1 Solutions

1) γ = 2 implies $\dfrac{1}{\sqrt{1-\beta^2}} = 2$, or $1-\beta^2 = \dfrac{1}{4}$. Thus $\beta = \dfrac{\sqrt{3}}{2} \approx 0.866$.

2) $$\frac{\gamma^2 - 1}{\gamma} = \frac{\frac{1}{1-\beta^2} - 1}{\gamma} = \frac{\frac{1-(1-\beta^2)}{1-\beta^2}}{\gamma} = \beta^2 \frac{\sqrt{1-\beta^2}}{1-\beta^2} = \gamma\beta^2 .$$

3) When β = 1 γ goes to infinity. The time dilation and length contraction formulas no longer make any sense.



4) a) When Bob observes Alice has moved 9×10^7m past the earth, he has timed her moving for time t given by $\frac{3}{5}ct = 9\times10^7$, or $t = \dfrac{5\cdot 9\times10^7}{3\cdot 3\times10^8} = 0.5s$. This is the time in Bob's frame. Alice's clock is recording proper time (it is in the same place on the ship), so it is slow by the factor $\gamma\left(\beta = \frac{3}{5}\right) = 5/4$. Thus Alice's clock reads 0.4s.

b) This is a different question. Alice observes Bob moving at $\frac{3}{5}c$, so a clock recording proper time in his frame runs slow. It reads $\dfrac{0.4s}{5/4} = 0.32s$.

The times are different because Alice and Bob disagree on simultaneity and on the positions of the events. (Alice's set of observers records Bob's clock at two different positions.)

Relativity: Classwork 2 solutions

3 December 2010

The train and tunnel frames agree that *t = 0* when the front of the train just enters the tunnel. Take the front of the tunnel to be *x* = 0. The train's speed is $\mathrm{v} = \tfrac{3}{5}c$.

1) Calculate the length of the tunnel according to the train: $L' = \frac{L}{\gamma} = \frac{100m}{5/4} = 80m$.

2) In terms of *L, v* and γ, write down expressions for the space-time coordinates of the following events in the tunnel frame:

a) The front of the train reaches the exit of the tunnel: $t_f = \frac{L}{\mathrm{v}}$, $x_f = L$.

b) The rear of the train reaches the entrance of the tunnel: The tunnel sees the train as length contracted, so $t_b = \frac{L}{\gamma \mathrm{v}}$, $x_b = 0$.

3) Use the Lorentz transformation equations to work out these space-time coordinates of these events in the frame of the train, again in terms of *L, v* and γ*:*

$x'_f = \gamma\left(L - \mathrm{v}\frac{L}{\mathrm{v}}\right) = 0$. In its frame the front of the train is defined to be $x'_f = 0$.

$t'_f = \gamma\left(\frac{L}{\mathrm{v}} - \frac{\mathrm{v}L}{c^2}\right) = \gamma\frac{L}{\mathrm{v}}\left(1 - \frac{\mathrm{v}^2}{c^2}\right) = \frac{L}{\gamma \mathrm{v}}$ (you could find this by direct reasoning).

$x'_b = \gamma\left(0 - \mathrm{v}\frac{L}{\gamma \mathrm{v}}\right) = -L$. In the train's frame the back is defined to be – *L*.

$t'_b = \gamma\left(\frac{L}{\gamma \mathrm{v}} - 0\right) = \frac{L}{\mathrm{v}}$. This is just the time for the rear of the tunnel to move the length of the train.

4) Calculate the times (in ns) t_f, t'_f, t_b, t'_b. $\mathrm{v} = \tfrac{3}{5}c$, $\gamma = \tfrac{5}{4}$. We find $t_f = 556ns$, $t_b = 444ns$, $t'_f = 444ns$, $t'_b = 556ns$.

5) Because the two events occur at different places, the two frames don't agree on their times. In the tunnel's frame the train is completely inside for (556 – 444)= 112*ns*. In the train's frame the front leaves the tunnel before the back arrives, so the back rocket clears the tunnel. Use the inverse Lorentz transform to find when the tunnel observes the train to fire its rocket: $t_{rocket} = \gamma\left(t'_f + \frac{\mathrm{v}x'_b}{c^2}\right) = \gamma\left(444ns + \frac{\tfrac{3}{5}c(-100m))}{c^2}\right) = 244ns$. This is before the front of the train even exits the tunnel, so the tunnel observers think the train has cheated!

Relativity: Classwork 3 solutions

15 December 2009

1) [λ_{red} = 630nm, λ_{green} = 530nm, $\lambda\nu = c$]. The Doppler shift formula is

$\frac{\lambda_0}{\lambda_D} = \sqrt{\frac{1+\beta}{1-\beta}}$ so $\left(\frac{\lambda_0}{\lambda_D}\right)^2 = \frac{1+\beta}{1-\beta}$. Thus $(1-\beta)\Delta^2 = 1+\beta$, where $\Delta = \frac{\lambda_0}{\lambda_D}$. Solving for β:

$$\beta = \frac{\Delta^2 - 1}{\Delta^2 + 1}$$. Δ = 630/530 = 1.189, so β = 0.171.

$$\frac{1}{\gamma}\frac{1}{1-\beta} = \frac{\sqrt{1-\beta^2}}{1-\beta} = \frac{\sqrt{(1-\beta)(1+\beta)}}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}}.$$

2) We know the separation is spacelike, so $-c^2\Delta t^2 + \Delta x^2 > 0$. The Lorentz transformation for time is $\Delta t' = \gamma\left(\Delta t - \mathrm{v}\Delta x / c^2\right)$, if $\Delta t' = 0$ then $c^2\Delta t = \mathrm{v}\Delta x$, or $\mathrm{v} = c^2\Delta t / \Delta x$. Note $\Delta x > c\Delta t$, so this velocity is less than *c*.

3) Express the following quantities in eV, MeV/c^2 etc. (c =3×10^{8}m/s, e = 1.6×10^{-19} C, 1eV = 1.6×10^{-19} J).

(a) Electron mass: m_e = 0.511 MeV/c^2.

(b) Proton mass: m_p = 938.3 MeV/c^2.

(c) Total energy of an electron with momentum p:

$E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{1^2 + 0.511^2}$ = 1.123 MeV/c^2.

(d) Kinetic energy of a proton with momentum p:

$$E - mc^2 = \sqrt{(pc)^2 + (mc^2)^2} - mc^2 \cong mc^2\left(1 + \tfrac{1}{2}\left(\frac{pc}{mc^2}\right)^2\right) - mc^2 \cong \tfrac{1}{2}\left(\frac{1}{938.3}\right)$$ = 0.5 keV/c^2.

(f) $\gamma_u = 1/\sqrt{1-(4/5)^2} = \frac{5}{3}$. Use $\gamma_u = E/mc^2$, so *E* = 1.56 GeV. *K* = 626 MeV.

4) $K = (\gamma_u - 1)mc^2$. a) 0 to 0.09c: *K* = 0.004 *mc*^{2}.

b) 0.9c to 0.99c: $\Delta K = (\gamma_{0.99} - \gamma_{0.9})mc^2$ = 7.09 – 2.29 = 4.8 *mc*^{2}.

There is a much larger mass to accelerate when the particle is already at 0.9c.

5) Your potential energy increases so your mass increases. $\Delta E = mgh = \Delta mc^2$, thus $\Delta m = mgh/c^2$ (this works because the change is mass is small). For m=60kg, Δm = 2×10^{-13}kg. This is small compared to the sweat you lose climbing the stairs!

6) $E = \gamma_u mc^2$, so $E - mc^2 = (\gamma_u - 1)mc^2$. Solve $(\gamma_u - 1)mc^2 = nmc^2$, or $\gamma_u - 1 = n$. Square both sides, $\gamma_u^2 = \frac{1}{1-\beta_u^2} = (n+1)^2$. Thus, $\frac{1}{(n+1)^2} = 1 - \beta_u^2$, or $\beta_u^2 = \frac{n^2+2n}{n^2+2n+1}$.

Finally, $u = c\sqrt{\frac{n^2+2n}{n^2+2n+1}}$. The point is that n can be large but the velocity changes slowly, and is always less than *c*.