

Relativity: Problem Sheet 1

Recall the length contraction formula is $l = l_0 / \gamma$, where l_0 is an object's length measured in a frame in which it is at rest. In class we derived this formula by considering the measurement of a moving rod in two frames. This problem leads you through an alternate derivation.

1) A lamp and a mirror are separated in the x direction by a distance l_0 in reference frame B. The time required for a light pulse to travel from the lamp to the mirror and back is thus

$$\Delta t_0 = \frac{2l_0}{c}.$$

Reference frame A observes the entire apparatus moving to the right with velocity v in the x direction. Frame A measures the separation to be l , which we want to determine in terms of l_0 and v . (You might want to draw a diagram to illustrate this.)

Call the time for the light to travel from the lamp to the mirror in frame A Δt_1 . How far does the mirror move in this time? What distance does the light pulse travel in terms of Δt_1 , v , and l ?

This distance must equal $c\Delta t_1$. You should therefore find that

$$\Delta t_1 = \frac{l}{c - v}.$$

Using the same logic, find the time Δt_2 for the light pulse to return from the mirror to the lamp in frame A.

The clock in frame B measures proper time because the departure and arrival of the light pulse are recorded in the same place. We therefore know that the total times in the two frames are related by the time dilation formula:

$$\Delta t_1 + \Delta t_2 = \gamma \Delta t_0.$$

Rewrite this equation in terms of l , l_0 , v and c . You should be able to show that

$$l = \frac{l_0}{\gamma}.$$

Recall that $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$.

2) [Young and Freedman 37.49] After being produced in a particle collision, a positive pion must travel down a 1.2km beam tube to reach an experimental area. A π^+ particle has a lifetime in its rest frame of $2.6 \times 10^{-8} s$. How fast must the π^+ travel if it is not to decay before it reaches the end of the tube? Since the velocity is very close to c , write $v = (1 - \epsilon)c$ and give your answer in terms of ϵ .

Relativity: Problem Sheet 2

1) It seems obvious that we can find the inverse Lorentz transformations by swapping the primes and reversing the sign of v , that is,

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx/c^2)$$

go to

$$x = \gamma(x' + vt'), \quad t = \gamma(t' + vx'/c^2).$$

Show this is true by explicitly solving the first set of equations for x and t .

2) The quantity $I = -(ct)^2 + x^2 + y^2 + z^2$ is called the “invariant interval” because it is unchanged under Lorentz transformations. In other words, the distance between an event and the origin changes and the time changes, but I doesn't. Prove this is true by applying a Lorentz transformation to I .

3) If an object moves with a velocity $u_x \leq c$ in one frame, prove that its velocity is always less than c in any other frame, assuming $v \leq c$.

4) A set of earth observers see a rocket ship moving to the right with an x velocity $u_R = \frac{2}{3}c$ and a second ship moving to the left with an x velocity $u_L = -\frac{4}{5}c$. What do the earth observers conclude about the relative velocity of one ship with respect to the other? If you find it is greater than c , is this a problem (in light of your answer to problem 3)? What does each ship conclude about the velocities of the earth and of the other ship?

5) We derived the velocity addition formula as for a particle moving with x -velocity u_x in a frame moving at velocity v as

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2}.$$

a) Show this result follows from the Lorentz transformations. That is, given velocity is defined as $u_x = \frac{dx}{dt}$, work out $u'_x = \frac{dx'}{dt'}$ from $dx' = \gamma(dx - vdt)$, $dt' = \gamma(dt - vdx/c^2)$.

It is okay to apply the Lorentz transformations to derivatives because they are linear. There are two events which define velocity: (x_1, t_1) particle leaves point 1,

(x_2, t_2) particle arrives at point 2. Velocity is just $u_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{dx}{dt}$.

b) Use the same method to work out the transformation rules for perpendicular velocities, $u_y = \frac{dy}{dt}$ and $u_z = \frac{dz}{dt}$.

Relativity: Problem Sheet 3

1) A pion at rest decays into a muon and a neutrino. Find the momentum of the outgoing muon, in terms of the two masses m_π and m_μ ($m_\nu = 0$). Use the notation

$$E_{\text{before}} = m_\pi c^2, \quad E_{\text{after}} = E_\mu + E_\nu, \quad p_{\text{before}} = 0, \quad p_{\text{after}} = p_\mu + p_\nu.$$

Energy and momentum are, of course, conserved. You will need to use the formula

$$E^2 = (pc)^2 + (mc^2)^2.$$

2) A particle of mass m whose total energy E_0 is twice its rest energy collides with an identical particle at rest. If they stick together, what is the mass M of the resulting composite particle? Show its velocity is $c/\sqrt{3}$.

3) The LHC will eventually be able to accelerate protons to 7TeV (1TeV = 10^{12} eV). What velocity would a tennis ball ($m = 58\text{gm}$) need to equal the kinetic energy of a 7TeV proton?

4) The first particle physics experiments involved accelerating a beam of protons to high energy E^* and then colliding them with a target at rest. This is simple, but an inefficient way to make new particles because some of the initial energy goes into the kinetic energy of the collision products, rather than into their mass. Classically, colliding a beam of energy E with a target at rest gives $\frac{1}{4}$ the centre of mass energy as colliding two beams of energy E .

a) Why is the factor $\frac{1}{4}$ classically? (Consider both particles to have the same mass.)

b) This is not a huge loss. The actual, relativistic, loss is enormous. Let E^* be the energy a single beam would need to have to equal the centre of mass energy of one of two colliding beams of energy E . Assuming the two particles have the same mass M use the Lorentz transforms to show that

$$E^* = \frac{2E^2}{Mc^2} - Mc^2.$$

c) When it is fully functional the LHC will collide protons, $M = 1\text{GeV}$ with $E = 7\text{TeV}$. What is E^* ? What multiple of E would this amount to? This explains why modern accelerators always need to face the huge technical challenge of colliding two beams.

5) [Young and Freedman, 37.68] A police radar measures the speed of a car by sending out a microwave beam with frequency f_0 and measuring the frequency shift of the reflected wave Δf . If the fractional frequency shift is $\Delta f/f_0 = 2.86 \times 10^{-7}$, what is the car's speed in km/h? (Hint: Are the waves Doppler shifted a second time when reflected off the car?)

6) The hydrogen atom emits bright red light at a wavelength $H_\alpha = 656\text{nm}$. The H_α lines measured on earth from opposite ends of the sun's equator differ by 0.009nm. Assuming the effect is caused by the rotation of the sun, find the period of rotation in days. The diameter of the sun is $1.4 \times 10^6\text{km}$. [answer: 25d; Use binomial expansions!]

7) [Young and Freedman, 37.70] A satellite moving at constant speed u relative to a ground receiver broadcasts a radio signal at constant frequency f_0 . As it approaches, the receiver measures a higher frequency f , as it recedes a lower frequency. When the satellite is directly perpendicular, it is neither moving towards or away from the receiver but there is still a frequency shift. Derive this shift and compare it to the relativistic shift for a source moving towards a receiver. (Hint: Successive wave crests move the same distance to the receiver and so they have the same transit time. Therefore $f = 1/T$. Use time dilation to relate the periods in the stationary and moving frames.) This shift is generally called the transverse Doppler shift.

8) Draw a space-time diagram representing a game of catch (or a conversation) between two people 10m apart. How is it possible for them to communicate, given that their separation is spacelike?

Relativity: Problem Sheet 1 Solutions

1) Call the time for the light to travel from the lamp to the mirror in frame A Δt_1 . How far does the mirror move in this time? $v\Delta t_1$. What distance does the light pulse travel in terms of Δt_1 , v , and l ? $d = l + v\Delta t_1$.

$$c\Delta t_1 = l + v\Delta t_1. \text{ Thus } \Delta t_1 = \frac{l}{c-v}.$$

On the way back, the lamp is moving towards the mirror, thus the distance is shorter, $d' = l - v\Delta t_2$. Therefore $\Delta t_2 = \frac{l}{c+v}$.

Given

$$\Delta t_1 + \Delta t_2 = \gamma \Delta t_0,$$

we have

$$\frac{l}{c-v} + \frac{l}{c+v} = \gamma \frac{2l_0}{c}.$$

Factor $1/c$ out of the left hand side and cancel this with the $1/c$ on the right. We have

$$\frac{l}{1-\beta} + \frac{l}{1+\beta} = 2\gamma l_0.$$

Solve for l :

$$l \left(\frac{1}{1-\beta} + \frac{1}{1+\beta} \right) = l \left(\frac{1+\beta+1-\beta}{(1-\beta)(1+\beta)} \right) = \frac{2l}{1-\beta^2} = 2l\gamma^2 = 2\gamma l_0.$$

$$\text{Thus } l = \frac{l_0}{\gamma}.$$

2) [Young and Freedman 37.49] After being produced in a particle collision, a positive pion must travel down a 1.2km beam tube to reach an experimental area. A π^+ particle has a lifetime in its rest frame of $2.6 \times 10^{-8} s$. How fast must the π^+ travel if it is not to decay before it reaches the end of the tube? Since the velocity is very close to c , write $v = (1 - \epsilon)c$ and give your answer in terms of ϵ .

Using length contraction, the pions see the tube as $L = \frac{1200m}{\gamma}$ long. The time it

takes to pass them is then $ct = \frac{L}{\beta} = \frac{1200m}{\beta\gamma}$. Setting t to be the lifetime gives the

minimum velocity, $\beta\gamma = \frac{1200m}{2.6 \cdot 3m} = 153.8$. Using the hint, $\beta = 1 - \epsilon$ and

$$\gamma = \left(1 - (1 - \epsilon)^2 \right)^{-1/2}. \text{ We find } \frac{1}{\sqrt{2\epsilon}} = 153.8, \text{ or } \epsilon = 2.11 \times 10^{-5}.$$

Relativity: Problem Sheet 2 Solutions

1) $x' = \gamma(x - vt)$ implies $\frac{x'}{\gamma} + vt = x$. Use this in $t' = \gamma(t - vx/c^2)$ so that

$$t' = \gamma \left(t - v \left(\frac{x'}{\gamma} + vt \right) / c^2 \right), \quad \text{or} \quad t' = t\gamma(1 - v^2/c^2) - vx'/c^2 = \frac{t}{\gamma} - \frac{vx'}{c^2}.$$

Solve for t : $t = \gamma(t' + vx'/c^2)$. The equation for x follows in the same way.

2) $I = -(ct)^2 + x^2 + y^2 + z^2$, $I' = -(ct')^2 + x'^2 + y'^2 + z'^2$. Since $y = y'$ and $z = z'$, the problem boils down to showing $-(ct')^2 + x'^2 = -(ct)^2 + x^2$. Write out left hand side:

$$-(c\gamma(t - vx/c^2))^2 + \gamma^2(x - vt)^2 = -\gamma^2(ct - \beta x)^2 + \gamma^2(x - \beta ct)^2.$$

Write out squares: $\gamma^2(-\tau^2 + 2\tau\beta x - \beta^2 x^2 + x^2 - 2\beta\tau x + \beta^2 \tau^2) = \gamma^2(-\tau^2[1 - \beta^2] + x^2[1 - \beta^2])$, where $\tau \equiv ct$.

$$[1 - \beta^2] = 1/\gamma^2, \text{ so we're done, } -(ct')^2 + x'^2 = -(ct)^2 + x^2.$$

3) $u_x \leq c$, but let us assume there is some v such that $u'_x > c$. This implies

$$\frac{u - v}{1 - uv/c^2} > c, \text{ which means } \frac{u}{c} - \frac{v}{c} > 1 - \frac{uv}{c^2}. \text{ Solve for } u: \frac{u}{c} \left(1 + \frac{v}{c} \right) > 1 + \frac{v}{c}, \text{ or } \frac{u}{c} > 1.$$

This contradicts our initial assumption, so $u'_x > c$ only if $u_x > c$.

4) The relative velocity of one ship with respect to the other is $\frac{2}{3}c + \frac{4}{5}c = \frac{22}{15}c$, but nothing physical is moving this fast, so there is no contradiction to 3). Ship L sees the earth moving at $v = \frac{4}{5}c$. In this case call the earth the prime frame, in which ship R is

moving at $u' = \frac{2}{3}c$. Thus ship L finds $u_R = \frac{\frac{2}{3}c + \frac{4}{5}c}{1 + \frac{2}{3} \cdot \frac{4}{5}} = \frac{22}{23}c$. According to ship R, the

earth is moving at $v = -\frac{2}{3}c$, thus $u_L = \frac{-\frac{4}{5}c - \frac{2}{3}c}{1 + \frac{2}{3} \cdot \frac{4}{5}} = -\frac{22}{23}c$, as you might expect.

$$5) \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx/c^2)}, \text{ so } \frac{dx'}{dt'} = \frac{(dx/dt - v)}{(1 - vdx/dt/c^2)} = \frac{u - v}{1 - uv/c^2}.$$

b) For y , $dy' = dy$, so $\frac{dy'}{dt'} = \frac{dy}{\gamma(dt - vdx/c^2)}$. Note the Lorentz transformation for time

involves the x coordinate, not y . Thus we find

$$\frac{dy'}{dt'} = \frac{dy/dt}{\gamma(1 - vdx/dt/c^2)} = \frac{u_y}{\gamma(1 - vu_x/c^2)}. \text{ For } z: u'_z = \frac{u_z}{\gamma(1 - vu_x/c^2)}.$$

Relativity

Problem Sheet 3 Solutions

1. Energy is conserved, so $E_\mu + E_\nu = m_\pi c^2$. For momentum $p_\mu = p_\nu$. We know $E_\mu = \sqrt{(p_\mu c)^2 + (m_\mu c^2)^2}$, $E_\nu = p_\nu c$. Thus

$$\sqrt{(p_\mu c)^2 + (m_\mu c^2)^2} + p_\mu c = m_\pi c^2 \rightarrow \sqrt{(p_\mu c)^2 + (m_\mu c^2)^2} = m_\pi c^2 - p_\mu c.$$

Solving for p_μ gives

$$p_\mu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}.$$

2. The total energy and total momentum are conserved. The total initial energy is $E_0 + mc^2$. The initial momentum is $pc = \sqrt{E_0^2 - m^2 c^4}$, entirely from the first particle. For the final particle, the energy-momentum relation is $E_f^2 = (pc)^2 + M^2 c^4$, so

$$(E_0 + mc^2)^2 = E_0^2 - m^2 c^4 + M^2 c^4 \rightarrow E_0^2 + 2E_0 mc^2 + m^2 c^4 = E_0^2 - m^2 c^4 + M^2 c^4.$$

Thus,

$$M^2 c^2 = 2m(E_0 + mc^2).$$

Given $E_0 = 2mc^2$, we have $M = \sqrt{6}m$. Now solve $E_f = \gamma_u M c^2 = 3mc^2$ or $\gamma_u = 3/\sqrt{6}$. Square both sides, $\gamma_u^2 = 3/2$, or $1 - \beta_u^2 = 2/3$ so $\beta_u = 1/\sqrt{3}$.

3. Ignoring the rest mass energy, convert 7TeV into joules: $7 \times 10^{12} \times 1.6 \times 10^{-19} = 1.12 \times 10^{-6} J$. Guessing the tennis ball is not relativistic, $\frac{1}{2}mv^2 = 1.12 \times 10^{-6} J$ gives $v = 6.2 mm/s$. This is a very large energy for a single proton.
4. a) An experiment with a beam with $E = \frac{1}{2}mv^2$ has total available energy E , since the particle at rest contributes no classical energy. In the center of mass frame, each beam has half the velocity, so the total energy is $2 \times \frac{1}{4}E = \frac{1}{2}E$. Colliding two beams of energy E gives a total energy $2E$, four times as much.
- b) In the frame where both protons have equal and opposite momentum, let E be the energy of one proton and p its momentum. We know $\beta_u = pc/E$ and $\gamma_u = E/mc^2$. We need to translate to a frame where one proton is at rest, which means a Lorentz transform with velocity u . The new energy E^* is

$$E^* = \gamma_u(E + \beta_u pc) \Rightarrow E^* = \frac{E}{mc^2} \left(E + \frac{p^2 c^2}{E} \right).$$

Thus $E^* = E^2/mc^2 + p^2/m$. Use $p^2 c^2 = E^2 - m^2 c^4$ to get $E^* = 2E^2/mc^2 - mc^2$.

- c) $E^* = 10^{17} eV = 10^5 TeV$. The increase in energy needed will be about 10^4 . This is impossible, as 7TeV is already the limit to the beam energy. The extra technical complications of a colliding beam experiment are well worthwhile.
5. There are two Doppler shifts (the car sees the source Doppler shifted and the police radar sees the car as a moving source). The total Doppler shift is thus the Doppler shift applied twice:

$$f_{2D} = f_0 \frac{1 + \beta}{1 - \beta}.$$

We are given the fractional frequency shift $\Delta f = (f_{2D} - f_0)/f_0$. This means

$$\Delta f = \frac{1 + \beta}{1 - \beta} - 1 = \frac{2\beta}{1 - \beta}.$$

We can ignore the denominator since the fractional shift is so small, find $v = 42.9m/s = 154km/h$.

6. Velocity at edge of rotating sun in $v = \pi D/T$, where D is the diameter and T is the period. One edge is shifted up in wavelength, the other down, so define

$$\lambda_+ = \lambda_0 \sqrt{\frac{1 + \beta}{1 - \beta}}, \quad \lambda_- = \lambda_0 \sqrt{\frac{1 - \beta}{1 + \beta}}.$$

Thus

$$\lambda_+ - \lambda_- = \lambda_0 \left(\sqrt{\frac{1 + \beta}{1 - \beta}} - \sqrt{\frac{1 - \beta}{1 + \beta}} \right).$$

Use $\sqrt{1 + x} \simeq 1 + \frac{1}{2}x$, etc. to find $\lambda_+ - \lambda_- \simeq 2\lambda_0\beta$. This gives a velocity $v = 2.06km/s$, which gives $T = 2.14 \times 10^6s$ which is about 25d.

7. $f = 1/T$. The satellite clock must be keeping proper time, so the earth observer measures a frequency $f_r = 1/\gamma T$ which differs by the factor $\sqrt{1 + \beta}$ from the Doppler shift when the source is moving directly towards the receiver. The transverse Doppler shift is due entirely to time dilation.
8. They can only communicate with a time delay. The events where one sends and the other receives a particular message are not spacelike separated.

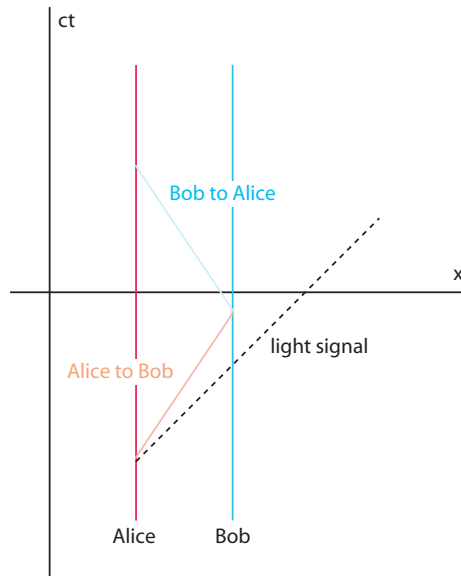


Figure 1: Alice and Bob at rest, communicating at less than the speed of light.