

Classwork I
Internal Combustion Engines

Information needed for this Classwork

- $PV = nRT$ ideal gas equation of state ($R \simeq 8.31 \text{ JK}^{-1}\text{mol}^{-1}$)
- For gas, $dW = -PdV$
- For adiabatic changes, $PV^\gamma = \text{constant}$, where for a fuel mixture $\gamma \simeq 1.4$
- At constant volume, $dQ = nC_{Vm} dT$

1. A petrol engine (Otto cycle) can be represented by a 6 process cycle (see figure below);
- In the intake stroke, the piston moves to allow the air/fuel mixture to enter the chamber at close to atmospheric pressure (isobarically).
 - The intake valve is then shut, and the mixture is compressed adiabatically.
 - Once compressed, the spark plugs fire to ignite the mixture. This results in an almost instantaneous rise in pressure at a constant volume (isochorically)
 - The heated fuel is allowed to expand adiabatically, until the chamber is once more at its maximum volume.
 - Whilst the chamber is at its maximum volume, the exhaust valve is opened and there is a decompression as hot gas leaves the chamber.
 - Finally, in the exhaust stroke, the remaining spent fuel is expelled with the exhaust valve open to atmospheric pressure. The exhaust valve is then shut and the intake valve opened in preparation for a new cycle.

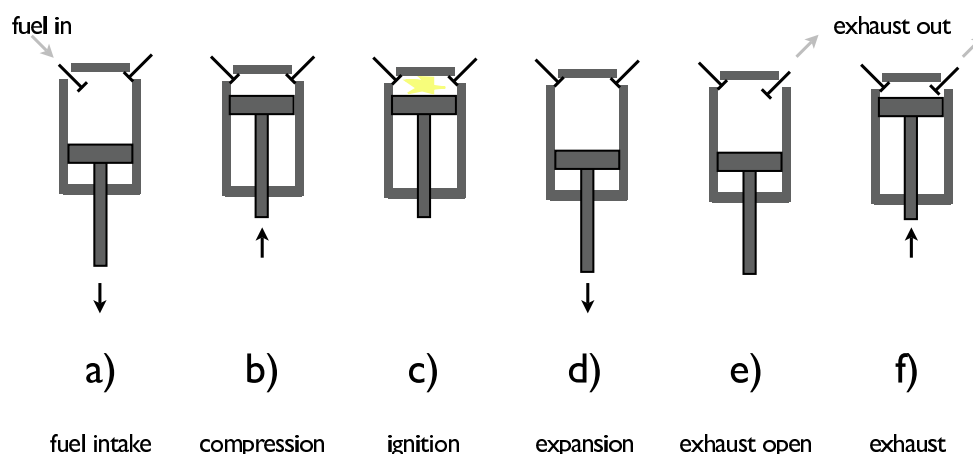


Figure 1: 6 stages of petrol engine

Draw this cycle on a PV diagram. What represents the work done by the engine per cycle on the diagram?

2. If the maximum volume of the cylinder $V_0 = 0.5$ litres ($= 0.5 \times 10^{-3} \text{ m}^3$), if it is filled at atmospheric pressure ($\sim 10^5 \text{ Pa}$) and 27°C . Calculate the number of moles of fuel mixture, assuming it is all gaseous.
3. Use the adiabatic relation $PV^\gamma = c$ and the ideal gas equation of state to find a relation between the volume and temperature in an adiabatic process. Hence for a compression ratio of 10:1, calculate the pressure and temperature (P_1, T_1) after the compression phase.
4. Show that the work done *on* a gas during an adiabatic compression from a state with temperature T_0 to one with temperature T_1 is given by,

$$\Delta W = \frac{nR}{\gamma - 1}(T_1 - T_0)$$

5. Calculate (numerically) the work done by the compression detailed in part 3. Comment on the sign of the work.
6. If the mixture's average molecular mass is $A^* \simeq 35$ and has a molar heat capacity $C_{Vm} = \frac{5}{2}R$, and the energy of combustion is $H_c \simeq 500 \text{ kJ kg}^{-1}$, calculate the temperature change on combustion. (Assume the number of moles of gas remains constant).
7. Calculate the temperature change and hence work done on expansion. What is the total work done by the gas during the cycle.
8. If an engine with four such cylinders is running at around 3000 revolutions per minute, calculate the power output. (You may want to quote this in horsepower, $1 \text{ hp} \approx 746 \text{ watts}$).
9. A diesel engine works in almost the same way as a petrol engine, except that the fuel is compressed to much higher pressure and temperature so that it self-ignites instead of requiring a spark for combustion. This burn phase can be represented as an isobaric expansion at the same final pressure as attained in the petrol engine. Include the PV cycle for a diesel engine of the same volume as your petrol engine on the same diagram. Which would be more efficient? Can you say why petrol engines are more common in cars?

Numerical Answers:

2) 0.02 moles 3) $2.5 \times 10^6 \text{ Pa}$, 754 K 5) 189 J 6) 840 K 7) -960 K, -399 J, 210 J 8) 21 kW, 28 hp

Classwork II
The Isothermal Atmosphere

Information needed for this Classwork

- Probability that a given molecule (mass m) is between heights z and $z + dz$ in an isothermal atmosphere (temperature T): $(mg/k_B T)e^{-mgz/k_B T} dz$.
 - Boltzmann's constant: $k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$.
 - Absolute zero = -273°C .
 - Acceleration due to gravity: $g = 9.81 \text{ m s}^{-2}$.
 - Radius of earth $r_E = 6,378 \text{ km}$
 - Height of Mount Everest = 8.8 km
 - For air, adiabatic constant $\gamma = 1.4$ and average molecular mass $m = 29 \times 1.66 \times 10^{-27} \text{ kg}$
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1. The probability that a given molecule in an isothermal atmosphere is between heights z and $z + dz$ can be written $p(z)dz$, where $p(z) = \frac{1}{\lambda}e^{-z/\lambda}$. Write down an expression for the 'scale height' λ in terms of T , the temperature, and m , the mass of a molecule.
2. The probability that a molecule is located between $z = z_1$ and $z = z_2$ is $\int_{z_1}^{z_2} p(z) dz$.
 - (a) Calculate the probability that a given molecule is between $z = 0$ and $z = \lambda$.
 - (b) Calculate the probability that a given molecule is between $z = \lambda$ and $z = 2\lambda$.
3. Show that $\int_0^\infty p(z) dz = 1$, and interpret this result.
4. Calculate λ for $T = 20^\circ \text{C}$ and $m = 4.82 \times 10^{-26} \text{ kg}$ (the average mass per molecule in air).
5. There is a 99.9% probability that any given molecule is located below a certain height z^* . Calculate the value of z^* for the parameters given in Q 5.
6. Write down a corresponding expression for the pressure as a function of height using this model, assuming atmospheric pressure at ground level is 1 bar. If humans can survive at pressures down to 0.5 bar, calculate the maximum height for human habitation.
7. In deriving the isothermal atmosphere model we assumed that g , the acceleration due to gravity, was independent of height. Was this a reasonable assumption?

- 8*. Of course, the earth's atmosphere is not isothermal. Better (though still not perfect) would be to assume an adiabatic relation; i.e. that as air rises, it does not exchange heat with neighboring air and so cools. Starting with the ideal gas equation of state, the first law for an adiabatic process and the theorem of equipartition of energy, show that the following relationship holds; $dP = \frac{\gamma}{\gamma - 1} \left(\frac{P}{T} \right) dT$
- 9*. Use this and the expression for the pressure differential for a slab in equilibrium to construct a differential equation for the rate of change of temperature with height. Hence show that the *adiabatic lapse rate* (the rate at which temperature falls as a function of height) is $\approx 9.8^\circ\text{K km}^{-1}$.
- 10*. What would be the temperature at the top of Mount Everest, assuming a temperature of 20°C at ground level. By solving the differential equation in part 9*, find the pressure at this height too, and compare it to the value obtained from the isothermal model.

Numerical Answers

2a) 0.63, b) 0.23; 4) 8.55 km; 5) 59.1 km 6) 5.9 km 10) $P_{adiab} = 0.29$ bar; $P_{isoth} = 0.35$ bar

Classwork III Derivation of the Ideal Gas Equation of State

In Lecture 1 we derived the ideal gas equation of state by considering gas molecules undergoing elastic collisions with a wall of the container and imparting momentum to it. The pressure is equal to the rate of momentum transfer per unit area. However, one aspect of that derivation was highly unsatisfactory. In this Classwork we do it properly.

Consider a box of volume V containing N molecules. Define the x axis pointing into the wall, as shown. The dodgy part of the previous derivation was the assumption that all the molecules had the same value of $|v_x|$, and then at the end replacing $|v_x|^2$ with the average value without any real justification. But we know that the molecules have values of v_x between $-\infty$ and $+\infty$, according to the velocity component distribution function: $f(v_x) = Ae^{-\alpha v_x^2}$ where

$$A = \left(\frac{m}{2\pi k_B T} \right)^{1/2} \quad \text{and} \quad \alpha = \frac{m}{2k_B T}.$$

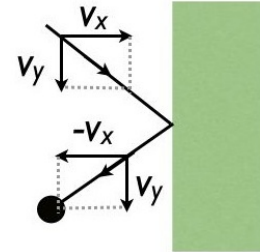


Figure 1: a collision

1. Write down an expression for the total number of molecules in the volume V with v_x between v_x and $v_x + dv_x$, and, hence, show that the number density of such molecules (the number per unit volume) is $n f(v_x) dv_x$ where $n = N/V$.
2. Considering particles with $0 < v_x < +\infty$ (molecules with $v_x < 0$ are moving away from the wall and won't hit it), show that the number of impacts with area L^2 of the wall¹ in time Δt by molecules with v_x between v_x and $v_x + dv_x$ is $dN_{coll} = n v_x f(v_x) dv_x \Delta t L^2$.
3. In Sec. 1.1 we found $N_{coll} = \frac{1}{2} L^2 |v_x| n \Delta t$. Why doesn't the factor $\frac{1}{2}$ appear in the new equation for dN_{coll} , in Q. 2?
4. By integrating the expression for dN_{coll} in part 2 (using the considerations in part 3 to choose appropriate limits), show the rate that particles hit the wall per unit surface area is given by $\frac{1}{4} n \langle v \rangle$, where $\langle v \rangle \equiv \left(\frac{8k_B T}{\pi m} \right)^{1/2}$ is the mean speed.
5. Each impact imparts momentum $2mv_x$ to the wall. Show that the contribution to the pressure on the wall from the molecules between v_x and $v_x + dv_x$ is $dP = 2nmv_x^2 f(v_x) dv_x$.
6. Using the standard integral $\int_0^\infty x^2 e^{-\alpha x^2} dx = \frac{1}{4} \left(\frac{\pi}{\alpha^3} \right)^{1/2}$ show that $P = nk_B T$ (the ideal gas equation of state).
7. It is tempting to write the pressure (\equiv total momentum change on the wall per unit area) to be the average rate at which particles strike the wall (part 4) multiplied by a change of momentum per collision of $2m\langle v \rangle$. Calculate this quantity (in terms of T). Can you give reasons why this is not the same as the expression given in part 6.

¹In Sec. 1.1 we considered an area A of the wall, but here we use the symbol A for the normalising constant in $f(v_x)$.

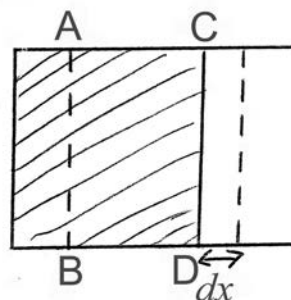
Classwork IV
Surface Tension

Information needed for this Classwork

- 1 atmosphere = $1.01 \times 10^5 \text{ N m}^{-2}$.
- Acceleration due to gravity: $g = 9.81 \text{ m s}^{-2}$
- Density of water: $\rho = 10^3 \text{ kg m}^{-3}$
- Surface tension of water: $\gamma = 0.0728 \text{ N m}^{-1}$

Liquids try to minimize their surface area; a liquid surface is like a stretched elastic membrane. The force responsible for this is *surface tension*.

Consider a soap film on a wire frame, as shown. The surface to the left of the imaginary line AB pulls the line to the left, but the surface to the right exerts an equal force to the right. The force acts perpendicular to the line and is proportional to the length, l , of the line; i.e. $F = \gamma l$, where γ is the surface tension (unit Nm^{-1}).



1. Assume section CD of the wire frame is movable. To keep it in place a force F_0 must be applied that is equal to the surface force pulling to the left. Increasing the applied force an infinitesimal amount dF will result in section CD moving an infinitesimal distance dx to the right. Show that the work done is $2\gamma l dx$. Hence, show that the increase in energy of the soap film can be written γdA , where dA is the change in the surface area. (Hint: remember a soap film has two surfaces.)
2. It requires energy γdA to increase the area of a fluid surface. You might think that at the microscopic (i.e. atomic) level this energy is used to pull the atoms further apart and increase their separation. But, in fact, the average interatomic separation doesn't change; liquids are incompressible, so the volume of the soap film is constant. The thickness of the film simply decreases when the area increases.

So, at the microscopic level, where does the energy go?

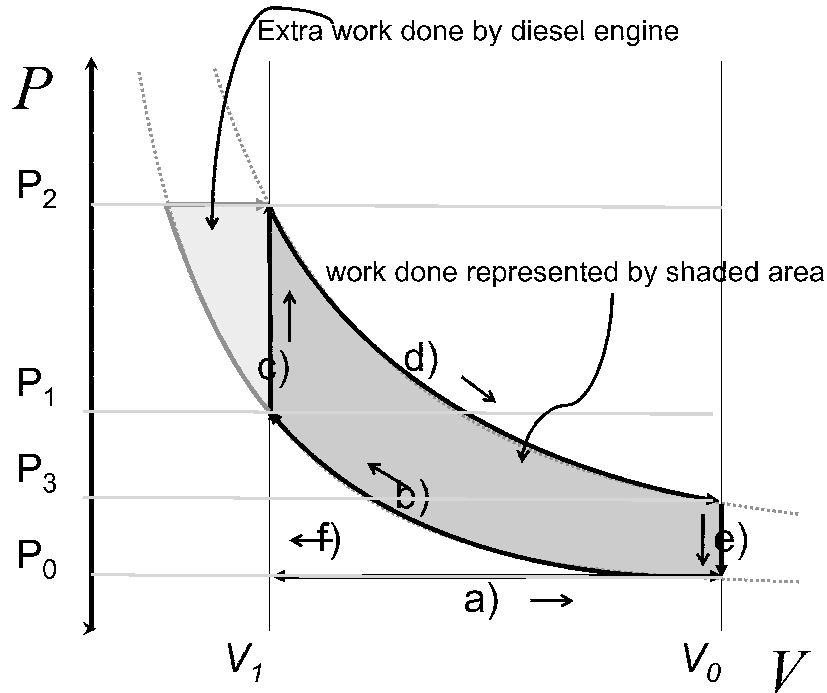
3. The surface of a spherical bubble (of radius r) in a liquid (e.g. fizzy drink) experiences a net inward force F_c towards the centre. This force can be found by calculating the work needed to increase the size of the bubble. If an infinitesimally larger outward force is applied to the surface the radius will increase by dr and the work done will be $F_c dr$. Find an expression for dA , the increase in surface area corresponding to an increase in radius of dr , and hence show that $F_c = 8\pi\gamma r$. (Hint: remember that this type of bubble, more properly called a *cavity*, has only one surface.)
4. In addition to this surface tension force, the surface is also acted on by P_{out} , the pressure of the surrounding liquid, and P_{in} , the pressure of the gas inside the bubble. Show that $P_{in} = P_{out} + \frac{2\gamma}{r}$. (This equation is called the Laplace equation.)
5. It is proposed to aerate a large open tank of water by pumping air into the water at a depth of 1 m below the surface. What air pressure is required to produce bubbles of radius 0.1 mm. Assume that the pressure at the surface of the water (i.e., at the top of the tank) is 1 atmosphere.
6. If a liquid is sufficiently pure (without bubbles or contaminants), and stored in a very smooth container, it is possible to heat it to well above its boiling point before it begins to boil. By looking at the form of the Laplace equation, suggest why it is more difficult to bring a pure liquid to boil.

Numerical Answer

5) $1.12 \times 10^5 \text{ N m}^{-2}$

Classwork I - Answers
Internal Combustion Engines

1. PV diagram for Otto cycle:



2. From $PV = nRT$, $n = \frac{PV}{RT} = \frac{10^5 * 5 \times 10^{-4}}{8.31 * 300} \simeq 0.02$ moles

3. $P_1 V_1^\gamma = P_0 V_0^\gamma$

So $P_1 = P_0 \left(\frac{V_0}{V_1}\right)^\gamma = 1 \text{ atm.} * \left(\frac{V_0}{(V_0/10)}\right)^{1.4} = 10^{1.4} \text{ atm} = 25.1 \text{ atm.} \simeq 2.5 \times 10^6 \text{ Pa}$

Also $PV^\gamma = \left(\frac{nRT}{V}\right) V^\gamma$ so $TV^{\gamma-1} = \text{constant}$

$\rightarrow T_1 V_1^{\gamma-1} = T_0 V_0^{\gamma-1}$

$T_1 = T_0 \left(\frac{V_0}{V_1}\right)^{\gamma-1} = 300 \left(\frac{V_0}{(V_0/10)}\right)^{1.4-1} = 300 * 10^{0.4} \simeq 754 \text{ K}$

4. $\Delta W = \int_{P_0, V_0}^{P_1, V_1} -P dV$ but $P = CV^{-\gamma}$

So, $\Delta W = \int_{V_0}^{V_1} -CV^{-\gamma} dV = \left[\frac{-1}{1-\gamma} CV^{-\gamma+1} \right]_{V_0}^{V_1}$

$\rightarrow \Delta W = \left[\frac{1}{\gamma-1} CV^{-\gamma} V \right]_{V_0}^{V_1} = \left[\frac{1}{\gamma-1} PV \right]_{P_0, V_0}^{P_1, V_1}$

but for ideal gas, $PV = nRT$, so

$$\rightarrow \Delta W = \left[\frac{1}{\gamma - 1} nRT \right]_{T_0}^{T_1}$$

$$\rightarrow \Delta W = \frac{nR}{\gamma - 1} (T_1 - T_0)$$

(Alternatively starting from $dU = dQ + dW = dW$ for adiabatic process since $dQ = 0$,

$$\text{but } U = \frac{n_d}{2} nRT \rightarrow dU = dW = \frac{n_d}{2} nRdT \quad (= C_V dT)$$

$$\text{But } \gamma = \frac{n_d + 2}{n_d} \rightarrow n_d(\gamma - 1) = 2 \rightarrow \frac{n_d}{2} = \frac{1}{\gamma - 1}$$

$$\text{So } \Delta W = \frac{nR}{\gamma - 1} \Delta T = \frac{nR}{\gamma - 1} (T_1 - T_0)$$

$$5. \text{ Work done (from above)} = \frac{nR}{\gamma - 1} (T_1 - T_0) = \frac{0.02 * 8.31}{0.4} (754 - 300) = 189 \text{ J.}$$

The sign should be positive to indicate work is done on compressing the gas.

$$6. \text{ Mass of gas } M = mn10^{-3} \text{ kg} = 35 \times 10^{-3} * 0.02 = 7 \times 10^{-4} \text{ kg}$$

$$\text{So heat of combustion } \Delta Q = H_c * M = 500 \times 10^3 * 7 \times 10^{-4} = 352 \text{ J}$$

$$\text{But } \Delta Q = C\Delta T = C_{Vm} * n * \Delta T \quad (\text{since heating is at const. } V)$$

$$\therefore \Delta T = \frac{\Delta Q}{\frac{5}{2} Rn} = \frac{350}{2.5 * 0.02 * 8.31} = 840 \text{ K}$$

$$\text{and } T_2 = T_1 + \Delta T = 754 + 840 = 1594 \text{ K}$$

$$7. \text{ using } T_3 V_3^{\gamma-1} = T_2 V_2^{\gamma-1}, \text{ where } V_2 = V_1 = V_0/10 \text{ and } V_3 = V_0$$

$$\text{So } T_3 = T_2 \frac{V_0/10^{0.4}}{V_0} = T_2 10^{-0.4} = 634 \text{ K}$$

$$\text{So } \Delta T = 634 - 1594 = -960 \text{ K}$$

$$\text{and } \Delta W_3 = \frac{nR}{\gamma - 1} (T_1 - T_0) = \frac{0.02 * 8.31}{0.4} (-960) \simeq -399 \text{ J}$$

(NB negative this time, since work done by gas as it expands, and so negative work done on it!)

$$\text{Since no work is done during isochoric processes, total work done} = \Delta W_1 + \Delta W_3 = 210 \text{ J}$$

(NB now +ve since work done *by* engine = work done *by* gas).

$$8. \text{ This is the work done in one cylinder in one cycle, but for four cylinders and each cycle takes 2 revolutions (think about how many compression and expansions are required per cycle), so rate of cycles} = 0.5 * 3000 \text{ rpm} = 25 \text{ Hz}$$

$$\text{Power output} = \Delta W * 25 * 4 = 210 * 100 \text{ watts} = 21.0 \text{ kW} = 28 \text{ hp}$$

$$9. \text{ Diesel engine gives greater shaded area and } \therefore \text{ greater work per cycle, and so is more efficient (greater work for same amount of heat of combustion).}$$

However, since cycle takes longer, diesel engines cannot be revved so fast, and therefore typically produce less power for the same size. Also to withstand higher compressions, they have to be heavier, and are often restricted to heavier vehicles (like trucks...).

Classwork II - Answers
The Isothermal Atmosphere

1. Comparing $(mg/k_B T)e^{-mgz/k_B T} dz$ and $p(z) = \frac{1}{\lambda} e^{-z/\lambda}$ gives $\lambda = \frac{k_B T}{mg}$

$$\begin{aligned} 2a. \int_0^\lambda p(z) dz &= \left[-e^{-z/\lambda} \right]_0^\lambda \\ &= \left[1 - \frac{1}{e} \right] \\ &= 0.632 \end{aligned}$$

$$\begin{aligned} 2b. \int_\lambda^{2\lambda} p(z) dz &= \left[-e^{-z/\lambda} \right]_\lambda^{2\lambda} \\ &= \left[\frac{1}{e} - \frac{1}{e^2} \right] \\ &= 0.233 \end{aligned}$$

$$\begin{aligned} 3. \int_0^\infty p(z) dz &= \left[-e^{-z/\lambda} \right]_\lambda^\infty \\ &= -[0 - 1] = 1. \end{aligned}$$

Total probability = 1, since particle must be somewhere!

$$4. \lambda = \frac{k_B T}{mg} = \frac{1.38 \cdot 10^{-23} \times 293}{4.82 \cdot 10^{-26} \times 9.81} = 8.55 \text{ km}$$

$$\begin{aligned} 5. \int_0^{z^*} p(z) dz &= 0.999 = \left[1 - e^{-z^*/\lambda} \right] \\ \rightarrow e^{-z^*/\lambda} &= 0.001 \\ \rightarrow \frac{z^*}{\lambda} &= \ln(1000) \\ \rightarrow z^* &= \ln(1000)\lambda = 59.1 \text{ km} \end{aligned}$$

6. $P(z) = P_0 e^{-z/\lambda} = 1 e^{-z/\lambda}$ bar, so for $P(z) = 0.5$ bar

$$\rightarrow e^{-z/\lambda} = 0.5$$

$$\rightarrow \frac{z}{\lambda} = \ln 2$$

$$z = \ln 2 \times \lambda = 5.9 \text{ km}$$

7. The scale length (8 km) is small compared to the earth's radius (as is the distance at which 99.9% of the gas is contained) since $r_E \approx 6400 \text{ km}$

8*. Ideal gas EOS $\rightarrow PV = Nk_B T$ differentiate $V dP + P dV = Nk_B dT$

$$\rightarrow -P dV = V dP - Nk_B dT$$

But first Law for adiabatic process ($dQ = 0$); $dU = -P dV = \frac{n_d}{2} N k_B dT$
 where we have used the equipartition of energy for the last bit.

Equating, $V dP - N k_B dT = \frac{n_d}{2} N k_B dT$

$$V dP = \left(\frac{n_d}{2} + 1\right) N k_B dT$$

but $\gamma = \frac{n_d + 2}{n_d} \rightarrow n_d = \frac{2}{\gamma - 1} \rightarrow \left(\frac{n_d}{2} + 1\right) = \frac{\gamma}{\gamma - 1}$

So $\rightarrow dP = \left(\frac{\gamma}{\gamma - 1}\right) \frac{N k_B}{V} dT = \left(\frac{\gamma}{\gamma - 1}\right) \left(\frac{P}{T}\right) dT$ Q.E.D.

9*. In eqm. $dP = -\rho g dz = -m n g dz = -m g \left(\frac{P}{k_B T}\right) dz$ where we used $P = n k_B T$

Equating to the above; $\left(\frac{\gamma}{\gamma - 1}\right) \left(\frac{P}{T}\right) dT = -\frac{m g}{k_B} \left(\frac{P}{T}\right) dz$

$$dT = -\frac{m g}{k_B} \left(\frac{\gamma - 1}{\gamma}\right) dz = \alpha dz$$

So $T(z) = T(0) - \alpha z$ where $\alpha = \frac{m g}{k_B} \left(\frac{\gamma - 1}{\gamma}\right) \approx 9.8 \text{ K km}^{-1}$

10*. For 8.8 km; $T = 20 - 8.8 * 9.8 \approx -66^\circ\text{C}$

(Actually due to heat released by water vapour as it condenses, the lapse rate is actually about 2/3 of what we calculated, so the temperature would only be about -40°C)

From [9*]; $\frac{dP}{P} = \left(\frac{\gamma}{\gamma - 1}\right) \left(\frac{dT}{T}\right)$

Integrate: $\ln P = \left(\frac{\gamma}{\gamma - 1}\right) \ln T + \ln C$

$$\rightarrow P T^{\frac{\gamma}{1-\gamma}} = C$$

$$P = P(0) \left(\frac{T(0)}{T}\right)^{\frac{\gamma}{1-\gamma}} \approx P(0) \left(\frac{293}{206}\right)^{\frac{1.4}{0.4}} \approx 0.29 \text{ atmospheres.}$$

This compares to $P = P(0) \exp(-z/\lambda) = P(0) \exp(-8.5/8.8) = 0.355$

which is a little less (but isothermal model gives a good estimate).

Classwork III - Answers
Derivation of the Ideal Gas Equation of State

1. Total no. of particles with x -component of velocity between v_x and $v_x + dv_x$ is

$$dN = Nf(v_x)dv_x$$

So the number density of these particles $dn = dN/V = (N/V) f(v_x) dv_x = nf(v_x) dv_x$

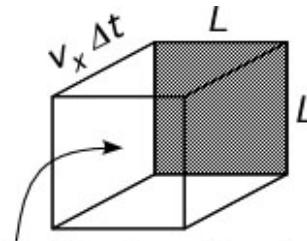
2. In time Δt particles with a x displacement of $v_x \cdot \Delta t$ from wall can reach it,

volume occupied by such particles which hit an area

$$L^2 = L^2 \cdot v_x \Delta t$$

and so number of such particles,

$$dN_{coll} = L^2 v_x \Delta t \times nf(v_x) dv_x$$



Particles from this volume with x velocity v_x will hit area L^2 in time Δt

3. We don't need extra $\frac{1}{2}$ since we are *only* going to consider particles travelling towards the wall by integrating between 0 and ∞ (i.e. forward going particles), even though v_x could have any value between $+\infty$ and $-\infty$
4. Rate at which particles with $v_x \rightarrow v_x + dv_x$ strike the wall per unit area;

$$R = \frac{1}{L^2} \frac{dN_{coll}}{dt} \approx dN_{coll}/L^2 \Delta T = nv_x f(v_x) dv_x$$

As above, integrating only over *forward* going particles, v_x ,

$$\begin{aligned} R &= \int_0^{\infty} Anv_x e^{-\alpha v_x^2} dv_x \\ &= nA \left[\frac{e^{-\alpha v_x^2}}{-2\alpha} \right]_0^{\infty} \\ &= \frac{n}{-2\alpha} \left(\frac{\alpha}{\pi} \right)^{1/2} [0 - 1] \\ &= \frac{n}{2} \left(\frac{1}{\pi\alpha} \right)^{1/2} \\ &= \frac{n}{2} \left(\frac{2k_B T}{\pi m} \right)^{1/2} \\ &= \frac{1}{4} n \left(\frac{8k_B T}{\pi m} \right)^{1/2} \\ &= \frac{1}{4} n \langle v \rangle \end{aligned}$$

5. momentum change of particles with v_x at the wall in time Δt is

$$\Delta p = 2mv_x \times dN_{coll} = 2mv_x \times L^2 v_x \Delta t \times n f(v_x) dv_x$$

So force on wall due to these particles is

$$dF = \Delta p / \Delta t = 2mv_x L^2 v_x \cancel{\Delta t} n f(v_x) dv_x / \cancel{\Delta t} = 2mv_x L^2 v_x n f(v_x) dv_x$$

$$\text{and pressure is given by } dP = dF / L^2 = 2mv_x \cancel{L^2} v_x n f(v_x) dv_x / \cancel{L^2}$$

$$\rightarrow dP = 2mnv_x^2 f(v_x) dv_x$$

$$6. P = \int dP = \int_0^\infty 2mnv_x^2 f(v_x) dv_x \quad (\text{again only forward going particles})$$

$$= \int_0^\infty 2mnv_x^2 A e^{-\alpha v_x^2} dv_x$$

$$= 2mnA \int_0^\infty v_x^2 e^{-\alpha v_x^2} dv_x$$

$$= 2mnA \frac{1}{4} \left(\frac{\pi}{\alpha^3} \right)^{1/2}$$

$$= 2mn \left(\frac{\alpha}{\pi} \right)^{1/2} \frac{1}{4} \left(\frac{\pi}{\alpha^3} \right)^{1/2}$$

$$= mn \frac{1}{2\alpha} = mn \frac{2k_B T}{2m}$$

$$\rightarrow P = nk_B T$$

$$7. \frac{1}{4} n \langle v \rangle * 2m \langle v \rangle = \frac{1}{2} nm \left(\frac{8k_B T}{\pi m} \right)$$

$$= \left(\frac{4}{\pi} \right) nk_B T \neq P$$

This does not take into account that only the v_x component of momentum changes (which is the reason that this is overestimated).

But even taking an average momentum change $\langle 2mv_x \rangle$ does not get this right, because it does not account for the fact that faster particles (which carry more momentum) collide with the walls more often than slower ones. Otherwise you might expect the answer of part 6 to be just $\frac{1}{6} n \langle v \rangle$, which is what you would get if you assumed all particles had the same velocity $\langle v \rangle$ but were directed equally between the 6 walls of the container.

Classwork IV - Answers
Surface Tension

1. $F_0 =$ force acting to left on CD $= 2 \times \gamma l$ NB γl from each surface

Work done $= F_0 dx = 2\gamma l dx$ (ignoring 2nd order term such as $dF \times dx$)

$dA =$ increase in surface area
 $= 2 \times l dx$ (each surface increases by $l dx$)

\rightarrow increase in energy = work done $= \gamma dA$

2. Remember that each atom in a liquid is bound by ~ 10 other atoms. But atoms at the surface only have ~ 5 nearest neighbours, with a corresponding breaking of ~ 5 bonds. By stretching the film, we increase the ratio of surface area to volume, so that the number of atoms near the surface has increased. Hence the work done has gone into breaking the bonds of this increasing number of surface atoms.

3. Surface area at radius r , $A = 4\pi r^2 \quad \rightarrow \quad dA = 4\pi \times 2r dr = 8\pi r dr$

\therefore work done $dW = \gamma dA = 8\pi\gamma r dr$

so $F_c = \frac{dW}{dr} = 8\pi\gamma r$

4. Total force on surface $= (P_{outside} \times 4\pi r^2) - (P_{inside} \times 4\pi r^2) + 8\pi\gamma r$

But total force $= 0 \quad \rightarrow \quad (P_{out} \times r) - (P_{in} \times r) + 2\gamma = 0$

$\rightarrow \quad P_{in} = P_{out} + \frac{2\gamma}{r}$

5. $P = P_0 + \rho g \delta$ (from lecture 10)

At depth $\delta = 1$ m, $P = 1.01 \times 10^5 + (10^3 \times 9.81 \times 1) = 1.11 \times 10^5 \text{ Nm}^{-2}$

So Pressure required $P_{in} = 1.11 \times 10^5 + (2 \times 0.0728/10^{-4}) = 1.12 \times 10^5 \text{ Nm}^{-2}$

6. To form a bubble (boil) the vapour pressure must be greater than the surface tension, but as seen by $F \propto (1/r)$, the surface tension is *greater* for a small bubble. Hence to form bubbles in a pure liquid is very difficult, since it requires the coming together of enough molecules with sufficient energy to evaporate. This inhibits boiling and allows the liquid to be superheated. If there are impurities (larger bubbles, or nucleation points), this reduces the surface tension allowing bubbles to form more easily. Once a bubble forms, as it grows the surface tension reduces, making it easier to grow further, and explaining the 'explosive' nature of boiling.