

Problem Sheet 1
Lectures 0–4

Learning Outcomes

Jargon

Macroscopic, microscopic, absolute temperature, mole, conduction, convection, radiation, internal energy, degrees of freedom, equilibrium, quasistatic process, isothermal, adiabatic, heat capacity, constant volume and constant pressure heat capacities, specific heat, molar specific heat.

Notation

K (kelvin), k_B , R , N_A , U , ΔU , ΔQ , ΔW , C_V , C_P , c_V , c_P , C_{V_m} , C_{P_m} , γ (ratio of specific heats).

Concepts

Qualitative differences between solids, liquids and gases at microscopic level; thermal motion; heat flow; ideal gas equation of state (and assumptions needed to derive it); relationship between P and U in ideal gas; theorem of equipartition of energy; qualitative distinction between work and heat at the microscopic level; first law of thermodynamics; calculating work done in quasistatic compression/expansion; plotting quasistatic process on PV diagram; process dependent nature of work and heat; relationship between heat capacities and number of degrees of freedom; relationship between P and V in adiabatic process.

Problems

- An ideal gas has a pressure of 1 atmosphere, a volume of 0.5 m^3 and a temperature of 300 K. Calculate the number of molecules it contains.
[1 atmosphere = $1.0 \times 10^5 \text{ N m}^{-2}$.]
 - Three moles of an ideal gas is at a pressure of 10^2 N m^{-2} and has a temperature of -10° C . Calculate its volume.
 - Calculate the volume occupied by 1 mole of an ideal gas at standard temperature and pressure (STP), i.e. 273.15 K and $1.01 \times 10^5 \text{ Pa}$ (a useful number to remember!)
 - Some regions of interstellar space is made of lone hydrogen atoms with a density of 1 particle / cm^3 , at a temperature of around 3 K. Calculate the pressure due to these particles.
- The plasma in a fusion reactor can be thought of as a mixture of two gases, an ion gas and an electron gas, both of which have a number density of $5.0 \times 10^{19} \text{ particles m}^{-3}$ and a temperature of 10^8 K . The volume of the plasma is 10^3 m^3 . Assuming that both ion and electron gases can be treated as ideal gases with three degrees of freedom, calculate the pressure and internal energy of each gas separately. Hence find the pressure and internal energy of the whole plasma. Express the plasma pressure in atmospheres.
- A monatomic ideal gas, initially at a pressure of 1 atmosphere, a volume of 0.5 m^3 , and a temperature of 300 K, goes through the following four part quasistatic cycle:
 - increase of pressure at constant volume to 1.5 atmospheres.
 - expansion at constant pressure to 1 m^3 ,
 - reduction of pressure at constant volume to 1 atmosphere,
 - compression at constant pressure to 0.5 m^3 ,

- (a) Plot these four parts on a single PV diagram.
- (b) Calculate ΔT , the temperature change, for each part separately.
- (c) Calculate ΔU , the internal energy change, for each part separately.
- (d) Calculate the total internal energy change for the whole cycle.
- (e) Calculate ΔW , the work done on the gas, during each part separately.
- (f) Calculate the total work done *by* the gas over the whole cycle.
- (g) You should have found that after going through the whole cycle the internal energy of the gas is unchanged, but it has done a finite amount of work. Where has the energy for this come from?
- (h) How is the total work done by the gas indicated on the PV diagram of part (a)?
4. (a) Argon is a monatomic gas with an atomic mass of 40.0. A volume of 1 m^3 of Argon is at a temperature of 300 K, and a pressure of 10^5 N m^{-2} . Assuming it can be treated as an ideal gas, calculate the number of molecules, the number of moles, and the mass of the gas.
- (b) Calculate the constant volume and constant pressure heat capacities, C_V and C_P (in J K^{-1}), and their ratio C_P/C_V .
- (c) Calculate the constant volume and constant pressure specific heats, c_V and c_P (in $\text{J K}^{-1} \text{ kg}^{-1}$), and their ratio c_P/c_V .
- (d) Calculate the constant volume and constant pressure molar specific heats, denoted C_{V_m} and C_{P_m} , (in $\text{J K}^{-1} \text{ mol}^{-1}$), and their ratio C_{P_m}/C_{V_m} .
5. (a) Consider two blocks of iron, one of mass 5 kg and initial temperature 50°C , the second of mass 10 kg and initial temperature 0°C . They are placed in thermal contact and allowed to reach equilibrium at the same temperature. Assuming there are no thermal losses to the surroundings, calculate the final temperature.
- (b) Each breath a student takes in a room at temperature of 20° is heated to a temperature of 37° before being expelled. If an average lung is 0.50 litres and the specific heat of air is 1020 J/kg K ($\rho_{air} = 1.3 \times 10^{-3} \text{ kg/litre}$), what is the heat lost over an hour due to breathing if the respiration rate is 20 breaths per minute.
- (c) The same student has a skin temperature of 35° and an emissivity of close to 1. Calculate the heat flow by radiation assuming he is well covered except for his head which has a surface area of $5 \times 10^{-2} \text{ m}^2$. What is the heat flow due to radiation per hour?
- (d) Due to surface evaporation, convection and conduction, the heat loss per student is closer to 50 W. If 100 such students were to release this much energy in an lecture, how much would the air temperature change in one hour in a room of volume 4000 m^3 assuming there is no air conditioning?
6. (a) An adiabatic process is one in which there is no heat flow ($dQ = 0$). Recalling that the internal energy of an ideal gas is $U = \frac{n_d}{2} N k_B T$ (where n_d is the number of degrees of freedom), show that the first law of thermodynamics for an adiabatic process in an ideal gas can be written:

$$\frac{dT}{T} = -\frac{2}{n_d} \frac{dV}{V}.$$
- (b) Integrate this to show that such a process satisfies: $TV^{2/n_d} = \text{constant}$.
- (c) Show that the equation found in part (b) can be rewritten as $PV^\gamma = \text{constant}$, where $\gamma = C_P/C_V$ is the ratio of heat capacities (as in Q. 4). What is the value of γ for a monatomic gas?
- (d) A monatomic ideal gas initially has pressure P_0 and volume V_0 . It then undergoes isothermal expansion to volume $2V_0$. A second monatomic ideal gas (with the same value of N) also initially has pressure P_0 and volume V_0 . It undergoes an adiabatic expansion to $2V_0$. Sketch

both of these processes on the same PV diagram. Which curve is steeper? Which ends at the higher temperature?

7. (a) Use the first law of thermodynamics to show that the work done in an adiabatic process ($dQ = 0$) in an ideal gas is given by

$$\Delta W = C_V (T_1 - T_0)$$

where T_0 and T_1 are the initial and final temperatures.

- (b) Alternatively, the work done can be found by integrating the pressure with respect to volume. Do this for an adiabatic process in an ideal gas, recalling that $PV^\gamma = \text{constant}$ in such a process (previous question) to show that

$$\Delta W = \frac{1}{\gamma - 1} (P_1 V_1 - P_0 V_0)$$

where P_0 and V_0 are the initial pressure and volume, and P_1 and V_1 are the final pressure and volume.

- (c) Show that the expressions obtained in parts (a) and (b) agree.

8. (a) The speed of sound in a gas is given by $v_s = \sqrt{\frac{\gamma P}{\rho}}$ where γ is the ratio of specific heats in the adiabatic law [Q. 6 (c)] and ρ is the density (in kg m^{-3}) of the gas. Show that if the sound speed is measured to be v_s at some temperature T then the number of degrees of freedom of the gas molecules can be found from:

$$n_d = 2 \left(\frac{m v_s^2}{k_B T} - 1 \right)^{-1}$$

where m is the mass of a gas molecule.

- (b) The average mass per molecule in air is 4.82×10^{-26} kg. The speed of sound in air at 20°C is 344 m s^{-1} . Calculate n_d in air (round to the nearest integer).
- (c) Is the answer to part (b) what you would have expected?

Numerical answers overleaf

Numerical Answers

1. (a) 1.22×10^{25} molecules
(b) 65.6 m^3
(c) 0.0224 m^3
(d) $4 \times 10^{-17} \text{ Pa}$
2. $P_{ion} = P_{electron} = 6.9 \times 10^4 \text{ Pa}$
 $U_{ion} = U_{electron} = 1.03 \times 10^8 \text{ J}$
 $P_{total} = 1.37 \text{ atmospheres}$
 $U_{total} = 2.07 \times 10^8 \text{ J}$
3. (b) 150 K, 450 K, -300 K , -300 K
(c) $3.79 \times 10^4 \text{ J}$, $1.136 \times 10^5 \text{ J}$, $-7.64 \times 10^4 \text{ J}$, $-7.64 \times 10^4 \text{ J}$
(e) 0, $-7.57 \times 10^4 \text{ J}$, 0, $5.05 \times 10^4 \text{ J}$
(f) $2.52 \times 10^4 \text{ J}$
4. (a) 2.42×10^{25} molecules, 40.1 moles, 1.60 kg
(b) 500 JK^{-1} , 833 JK^{-1} , 1.67
(c) $312 \text{ JK}^{-1}\text{kg}^{-1}$, $521 \text{ JK}^{-1}\text{kg}^{-1}$, 1.67
(d) $12.5 \text{ JK}^{-1}\text{mol}^{-1}$, $20.8 \text{ JK}^{-1}\text{mol}^{-1}$, 1.67
5. (a) 16.7°C
(b) 13,000 J
(c) 16,600 J
(d) 3.4°C
6. (c) $\gamma = 5/3$ for a monatomic ideal gas
8. (b) 5

Problem Sheet 2
Lectures 5 and 6

Learning Outcomes

Jargon

Isothermal atmosphere, distribution function, velocity component distribution, Maxwell speed distribution, moment of a distribution function, velocity space.

Notation

$$f(v_x), f(v), v_{mp}, \langle v \rangle, \langle v^2 \rangle, v_{rms}.$$

Concepts

Differential equation for the variation of pressure with height in an isothermal atmosphere (derivation, and solving it to find the variation of number density with height); Boltzmann law; normalizing distribution function; deriving the expression for v_{mp} in a Maxwell distribution; given appropriate standard integrals, be able to find $\langle v \rangle$ and $\langle v^2 \rangle$.

Problems

1. The velocity component distribution function has the form $f(v_x) = Ae^{-\alpha v_x^2}$ where $\alpha = m/2k_B T$ and A is a constant. Use the fact that a molecule must have *some* value of v_x between $-\infty$ and $+\infty$ to show that $A = \left(\frac{m}{2\pi k_B T}\right)^{1/2}$.
2. a) Show that the most probable speed in a Maxwell distribution is $v_{mp} = \left(\frac{2k_B T}{m}\right)^{1/2}$.
b) Find the variation of the maximum probability (i.e. the value of the distribution function at v_{mp}) as a function of temperature.
3. In Sec. 6.5 of the lectures we found that the average speed in a Maxwell distribution is given by $\langle v \rangle = 4\pi A^3 \int_0^\infty v^3 e^{-\alpha v^2} dv$. Without quoting a standard integral (i.e. integrating by parts), show that $\langle v \rangle = \left(\frac{8k_B T}{\pi m}\right)^{1/2}$.
4. The mean square speed, $\langle v^2 \rangle$, i.e., the average value of v^2 , is given by $\int_0^\infty v^2 f(v) dv$ (integrals of the form $\int_0^\infty v^n f(v) dv$ are called moments of the distribution function). Show that $\langle v^2 \rangle = \frac{3k_B T}{m}$, and hence write down an expression for the average kinetic energy of a monatomic molecule in a Maxwell distribution.
5. We can identify three characteristic speeds in a Maxwell distribution: (1) v_{mp} , (2) $\langle v \rangle$, (3) $v_{rms} = (\langle v^2 \rangle)^{1/2}$, i.e., the root mean square speed. Calculate the values of these three speeds for O_2 molecules at $20^\circ C$ (the atomic mass of Oxygen is 16.0).

6. In Problem Sheet 1 (Q. 2) we considered the plasma in a fusion reactor in which the ions and electrons both have a temperature of 10^8K . The ions are a mixture of deuterium and tritium. For each of the three types of particles in the plasma (deuterium ions, mass $3.34 \times 10^{-27}\text{ kg}$, tritium ions, mass $5.01 \times 10^{-27}\text{ kg}$, and electrons, mass $9.11 \times 10^{-31}\text{ kg}$) calculate
- the average particle speed,
 - the average particle energy in Joules and eV.
7. Show by integrating over the velocity component distribution function that $\langle v_x \rangle = 0$.
8. In Classwork II we wrote the probability of a molecule being between heights z and $z + dz$ in an isothermal atmosphere as $p(z)dz$, where $p(z) = \frac{1}{\lambda}e^{-z/\lambda}$, and $\lambda = \frac{k_B T}{mg}$. Show that the average height of a molecule in the atmosphere is λ , and hence obtain an expression for the average potential energy in terms of T . [Hint: this question involves an integral which is not on the list in Handout 1. You might consider doing it by parts.]

Numerical Answers

5. 390 ms^{-1} , 440 ms^{-1} , 478 ms^{-1} .
6. (a) Deuterium ions: $1.03 \times 10^6\text{ ms}^{-1}$,
 Tritium ions: $8.37 \times 10^5\text{ ms}^{-1}$,
 electrons: $6.21 \times 10^7\text{ ms}^{-1}$.
- (b) $2.07 \times 10^{-15}\text{ J}$ for all three types.

Problem Sheet 3
Lectures 7–9

Learning Outcomes

Jargon

Translational, rotational and vibrational degrees of freedom, van der Waals force, ionic and covalent bonds, van der Waals equation of state, mean free path.

Concepts

Degrees of freedom for a diatomic molecule; how quantum effects modify the number of degrees of freedom of real gases, and how the number varies as the temperature is raised for a diatomic gas; the Lennard-Jones 6-12 potential; the origin of the coefficients a and b in the van der Waals equation of state; understand qualitatively why the internal energy of a van der Waals gas differs from an ideal gas with the same N , V and T ; the form of the van der Waals isotherms on a P-V diagram; derivation of an expression for the mean free path, assuming rigid spherical particles, only one of which is moving.

Problems

1. Show that $\langle v_x^2 \rangle = \frac{k_B T}{m}$.
2. A diatomic molecule can be regarded as two identical particles joined by a spring. At any instant the x coordinates (defining the x direction as shown) of the centres of the two particles are $x_1 = x - d/2$ and $x_2 = x + d/2$, where x is the x coordinate of the centre of mass of the molecule and d is the separation of the particles. By writing $d = d_0 + \xi$, where d_0 is the equilibrium separation and ξ is the displacement from equilibrium, show that the kinetic energy associated with the x component of the two particles' velocities is $\frac{1}{2}mv_x^2 + \frac{1}{2}\left(\frac{m}{4}\right)\nu^2$ where m is the mass of the whole molecule, $v_x = \frac{dx}{dt}$ = x component of the centre of mass velocity, and $\nu = \frac{d\xi}{dt}$.
3. The Lennard-Jones 6-12 potential has the form $U(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$.
 - (a) Calculate the equilibrium separation r_0 in terms of A and B .
 - (b) Calculate the value of the potential at equilibrium $U(r_0)$
 - (c) Given that the binding energy equals $-\epsilon$ (i.e. $U(r_0) = -\epsilon$) show that the Lennard-Jones potential can be rewritten in the form, $U(r) = \epsilon \left\{ \left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6 \right\}$
 - (d) The angular frequency of oscillations of a particle in a van der Waal lattice about its equilibrium position is given by $\omega_E = \left(\frac{2}{m} \frac{d^2U(r)}{dr^2}\right)^{1/2}$ for $r = r_0$. (Remember the 2 comes from the fact that there are two bonds in any one direction). Calculate this frequency, and find the phase velocity of these oscillations for a wavelength $\lambda = r_0$, in terms of r_0 and ϵ . (This is the speed of sound in a VW solid.)
4. (a) We can generalize the Lennard-Jones 6-12 potential by assuming an attractive potential $\propto -r^{-n}$, while keeping the repulsive potential $\propto r^{-12}$. The potential energy of the interaction can be written $U = \frac{A}{x^{12}} - \frac{B}{x^n}$ where A and B are constants, $x = r/r_0$ and r_0 is the equilibrium separation. Show that $B = 12A/n$, and, hence, that the magnitude of the binding energy is given by $\epsilon = A \left(\frac{12}{n} - 1\right)$.

- (b) The van der Waals force corresponds to $n = 6$, while the Coulomb force responsible for ionic bonding corresponds to $n = 1$. Assuming that the repulsive force (i.e., the value of A) is the same in both cases, compare the depths of the van der Waals and ionic potential wells. What does this suggest about the relative strengths of van der Waals and ionic bonds?
5. A simple model of a gas treats the molecules as hard spheres of radius a , which only interact through collisions (i.e. there is no attractive force between them). Consider one molecule to be moving, and the others stationary. The moving molecule's path is deflected at each collision, but if we straighten it out we could imagine the molecule sweeping out a cylinder of cross-sectional area σ such that a collision will occur whenever the centre of one of the stationary molecules falls within the cylinder.
- Write down an expression for σ in terms of a .
 - Show that the average number of collisions experienced by the moving molecule in travelling a distance l is $4\pi nla^2$, where n is the density of molecules.
 - Hence show that the average distance travelled before experiencing a collision (the mean free path) is given by $\lambda_m = \frac{1}{4\pi na^2}$.
 - When the motion of the other molecules is taken into account the value of λ_m is reduced by a factor of $\sqrt{2}$. Assuming the ideal gas equation of state show that the mean free path can be written $\lambda_m = \frac{k_B T}{4\sqrt{2}\pi a^2 P}$.
 - Calculate λ_m for an oxygen molecule in air at 20°C and a pressure of 1 atmosphere ($= 1.01 \times 10^5 \text{ N m}^{-2}$). Assume the molecule is a sphere of radius $2.0 \times 10^{-10} \text{ m}$.
 - In Problem Sheet 2, we found that the average speed of an oxygen molecule at 20°C was 440 m s^{-1} . Calculate the average time between collisions.
6. Show that the constant volume heat capacity of a van der Waals gas is the same as that of a monatomic ideal gas containing the same number of molecules.
7. Assuming that Oxygen molecules are spheres of radius $2.0 \times 10^{-10} \text{ m}$, estimate the pressure at which the finite size of the molecules causes a 1% departure from ideal gas behaviour at 20°C . (Ignore the attractive force between molecules.)
8. In Section 9.1 of Lecture 9 we saw that the low temperature isotherms of the van der Waals gas on a $P - V$ diagram have a maximum and a minimum, while the high temperature ones don't. The dividing line between these two types is an isotherm along which there is a point (called the *critical point*) at which $\left(\frac{\partial P}{\partial V}\right)_T = 0$ and $\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$. The temperature of this isotherm is called the *critical temperature*, T_c . Show that $V_c = 3Nb$ and $T_c = \frac{8a}{27k_B b}$, where V_c is the volume at the critical point, and a and b are the constants in the van der Waals equation of state. Hence calculate the pressure P_c at the critical point.

Numerical Answers

4. (b) Ionic potential well is $11 \times$ deeper.
5. (e) $5.63 \times 10^{-8} \text{ m}$,
 (f) $1.28 \times 10^{-10} \text{ s}$.
7. About 3 atmospheres.

Problem Sheet 4
Lectures 9-11

Learning Outcomes

Jargon

Buoyancy, mass flow rate, isobar, isochor, critical point, triple point, melting, vaporization, sublimation, metallic bond, linear and volume thermal expansion coefficients, sound waves, bulk modulus Bose-Einstein condensate, ionisation, plasma.

Concepts

Pressure gradient force in fluid; Archimedes' Principle; continuity equation; classical expressions for the internal energy and constant volume heat capacity of a solid; thermal expansion; calculating the frequency of sound waves, calculating plasma frequency and Debye length.

Problems

- An iceberg of density 920 kg m^{-3} floats in seawater of density 1025 kg m^{-3} . What fraction of its volume is submerged?
 - To what radius must an approximately spherical balloon of mass 5g be filled with helium gas for it be able to support its own weight due to the buoyant force of the displaced air? (Assume both air and He are at STP).
- Water flows at 1.2 m s^{-1} through a hose-pipe of radius 0.8 cm. Calculate the speed with which it emerges from a nozzle of radius 0.4 cm.
 - How long would it take to fill a tank of volume 20 m^3 with this hose-pipe?
 - If the pressure of the water was 2 bar as it flows into the pipe, calculate its pressure at the exit (assuming it is kept horizontal).
- The van der Waals constants for Nitrogen are $a = 3.86 \times 10^{-49} \text{ J m}^3$ and $b = 6.49 \times 10^{-29} \text{ m}^3$.
 - Assuming that a Nitrogen molecule is spherical, estimate its radius.
 - Use the result of Q8, Problem Sheet 3, to estimate the critical temperature of Nitrogen. (For comparison the actual value is 126 K.)
- Using the law of equipartition of energy, write down the internal energy of a solid at room temperature. Show that the constant volume heat capacity of a solid is $C_v = 3Nk_B$. (This result is called the Dulong and Petit Law.)
 - Calculate the classical value of the molar specific heat of a solid in J K^{-1} , and also in calories K^{-1} . (1 calorie = 4.18 J)
 - 27 g of aluminium at 10° C is put in contact with 63.5 g of copper at 100° . Calculate their final temperature. (Atomic masses are $A_{Al} = 27$, $A_{Cu} = 63.5$).

5. A solid expands when its temperature is raised. A temperature increase of ΔT will produce an increase in the length of an object of $\Delta L = \alpha L_0 \Delta T$, where L_0 is the initial length and α is the *linear thermal expansion coefficient*.

- (a) Calculate the contraction of a 30 cm aluminium ruler from room temperature (20°C) when put in liquid nitrogen ($T_{\text{melt}} = -196^\circ\text{C}$) ($\alpha_{\text{Al}} = 23.1 \mu\text{m} \cdot \text{m}^{-1} \cdot \text{K}^{-1}$).
- (b) The Eiffel Tower is made of steel ($\alpha = 1.17 \times 10^{-5} \text{K}^{-1}$). At 20°C it has a height of 320 m. Calculate how much shorter it is at -10°C .
- (b) The *volume thermal expansion coefficient*, β , is defined such that a temperature increase ΔT produces a volume increase $\Delta V = \beta V_0 \Delta T$, where V_0 is the initial volume. Show that for small temperature changes $\beta = 3\alpha$.

6. A system has *only* two microstates with energies E and $E + \Delta E$ ($\Delta E > 0$). Assuming that the system is in thermal equilibrium;

- (a) If there are N indistinguishable particles in total in either of the two states, calculate the populations of both levels.
- (b) Calculate the relative populations when
 - i. $k_B T = 0.1 \Delta E$
 - ii. $k_B T = 0.3 \Delta E$
 - iii. $k_B T = \Delta E$
 - iv. $k_B T = 10 \Delta E$
- (c) The bond energy for a rubidium lattice is 0.023 eV per atom. Calculate the temperature at which the mean kinetic energy of a rubidium atom equals this energy. What would you expect to happen to rubidium at this temperature.
- (d) The first ionisation energy of rubidium is 4.2 eV. Calculate the temperature at which the kinetic energy of a rubidium atom is equal to this energy. What would you expect to happen to rubidium at this temperature.
- (e) Rubidium was the first substance to be made into a Bose-Einstein condensate. To make the condensate the de Broglie wavelength of the Rb atom must be greater than the mean spacing of the atoms $= n^{-1/3}$. Assuming that the mean kinetic energy is equivalent to $\frac{3}{2} k_B T$, calculate the mean wavelength of the condensate atoms λ . Hence show that,

$$T_c < \frac{h^2}{3mk_B} n^{2/3}$$
- (f) Calculate the temperature required to create a Bose-Einstein condensate of 2000 Rb atoms ($A = 86$) trapped in a cube with sides of length $10 \mu\text{m}$. What is the corresponding quantum of energy between the condensed and excited states?

7. Sound waves are compression waves. Regions of higher density (and thus pressure) expand, and exert a force on neighbouring regions (in a gas due to collisions), which causes it to compress and so the compression to propagate as a wave.

- (a) The speed of sound is defined as $c_s = \sqrt{\frac{\text{bulk modulus}}{\text{density}}}$, where the bulk modulus is defined as $K = -V \left(\frac{\partial P}{\partial V} \right)$

Assuming sound waves in a gas can be described by an adiabatic equation of state, show that $c_s = \sqrt{\gamma P/\rho}$.

- (b) Rewrite the sound speed in terms of temperature for an ideal gas. Hence calculate the sound speed in air ($\gamma = 1.4$), and in pure helium gas both at STP. (For air, the mean molecular mass $\bar{A} = 29.0$, and for helium $A = 4.0$).
- 8.* Consider an exactly neutral hydrogen plasma in which $n_e = n_i = n_0$ (n_e and n_i are the electron and ion number densities) and the charge density $\rho_q = (n_i - n_e)e$ is zero everywhere. If all the electrons in a layer of width δ are shifted distance δ in one direction a charged layer will be formed in the plasma.
- (a) Show that this sets up an electric field with a the maximum value of $E_{max} = \frac{n_0 e d}{\epsilon_0}$. (Hint set up Gauss' law as for a capacitor).
- (b) The space charge field set-up tries to restore the displaced charge sheet of electrons back to their original position. The equation of motion for the electrons in the charge sheet can be written,
- $$m_e \frac{d\delta}{dt} = -eE_{max}$$
- Show that this implies SHM for the charge sheet, with a frequency equal to the plasma frequency. Why can the motion of the ions be ignored?
- (c) The work done to move the electrons is stored in the electric field. Given that the energy density in an electric field is $u_E = \frac{1}{2}\epsilon_0 E^2$ (in J m^{-3}), write down an expression for the maximum value of u_E .
- (d) Small non-neutral regions can arise spontaneously in the plasma as a result of thermal fluctuations. The energy to form them comes from the translational kinetic energy of the electrons. The average energy per unit volume available to form the charged layer described above is $n_0 \times \frac{1}{2}k_B T$ (the relevant electron motion corresponds to one degree of freedom). Equating this to the maximum u_E gives an expression for δ_{max} , the maximum width of the layer. Show that $\delta_{max} = \left(\frac{\epsilon_0 k_B T}{n_0 e^2}\right)^{1/2}$ (= the Debye length).
- (e) Calculate the plasma frequency and Debye length in a magnetically confined fusion plasma in which $n_0 = 10^{20} \text{ m}^{-3}$ and $T = 10^8 \text{ K}$.

Numerical Answers

1. (a) 89.8%, (b) $r \approx 10 \text{ cm}$
2. (a) 4.8 ms^{-1} , (b) $8.3 \times 10^4 \text{ s}$ (c) 1.89 bar
3. (a) $1.57 \times 10^{-10} \text{ m}$, (b) 128 K.
4. (b) $24.9 \text{ JK}^{-1}\text{mol}^{-1}$, 6.0 cal/K, (c) 55°C
5. (a) 1.5 mm (b) 11.2 cm.
7. (b) 331 m/s, 973 m/s
8. (f) $5.6 \times 10^{11} \text{ Hz}$, $6.91 \times 10^{-5} \text{ m}$.

Structure of Matter

Problem Sheet 1 Answers

$$1. \quad (a) \quad N = \frac{PV}{k_B T} = \frac{0.5 \times 10^3}{1.38 \times 10^{-23} \cdot 300} = 1.21 \times 10^{25}$$

$$(b) \quad V = \frac{nRT}{P} = \frac{3 \times 8.31 \times 263}{10^2} = 65.6 \text{ m}^3$$

$$(c) \quad V = \frac{8.31 \times 273}{1.01 \times 10^5} = 0.0224 \text{ m}^3 = 22.4 \text{ litres} = 22,400 \text{ cc}$$

$$(d) \quad n = 1 \text{ cm}^{-3} = 10^6 \text{ m}^{-3}; \quad P = nk_B T = 10^6 \cdot 1.38 \times 10^{-23} \cdot 3 = 4.14 \times 10^{-17} \text{ Pa}$$

$$2. \quad P = Nk_B T / V = nk_B T, \text{ so } P_e = P_i = 5 \cdot 10^{19} \times 1.38 \cdot 10^{-23} \times 10^8 = 6.9 \cdot 10^4 \text{ Pa}$$

$$P_{Total} = P_e + P_i = 1.38 \cdot 10^5 \text{ Pa} = 1.37 \text{ atmospheres}$$

$$U_i = U_e = \frac{3}{2} Nk_B T = \frac{3}{2} PV = \frac{3}{2} \times 6.9 \cdot 10^4 \times 10^3 = 1.03 \cdot 10^8 \text{ J}$$

$$U_{Total} = U_i + U_e = 2.07 \cdot 10^8 \text{ J}$$

3. (a)

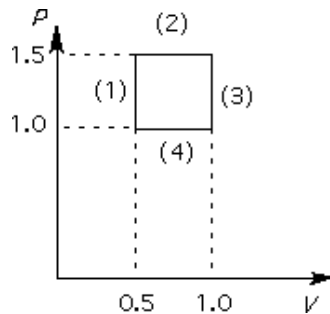


Figure 1: default

(b) Using $\frac{P_1 V_1}{T_1} = Nk_B = \text{constant}$, so

$$(1) \quad \frac{1 \cdot 0.5}{300} = \frac{1.5 \cdot 0.5}{T_1} \rightarrow T_1 = 450 \text{ K} \rightarrow \Delta T = 150 \text{ K}$$

$$(2) \quad \frac{1.5 \cdot 0.5}{450} = \frac{1.5 \cdot 1}{T_2} \rightarrow T_2 = 900 \text{ K} \rightarrow \Delta T = 450 \text{ K}$$

$$(3) \quad \frac{1.5 \cdot 1}{900} = \frac{1 \cdot 1}{T_3} \rightarrow T_3 = 600 \text{ K} \rightarrow \Delta T = -300 \text{ K}$$

$$(4) \quad \frac{1 \cdot 1}{300} = \frac{1 \cdot 0.5}{T_4} \rightarrow T_4 = 300 \text{ K} \rightarrow \Delta T = -300 \text{ K}$$

- (c) $\Delta U = \frac{3}{2}Nk_B(T_1 - T_0) = \frac{3}{2}(P_1V_1 - P_0V_0)$ so,
- (1) $\Delta U = \frac{3}{2}(1.5 * 1.01 \times 10^5 * 0.5 - 1.0 * 1.01 \times 10^5 * 0.5) = 4.13 \times 10^4 \text{ J}$
 - (2) $\Delta U = \frac{3}{2}(1.5 * 1.01 \times 10^5 * 1.0 - 1.5 * 1.01 \times 10^5 * 0.5) = 12.39 \times 10^4 \text{ J}$
 - (3) $\Delta U = \frac{3}{2}(1.0 * 1.01 \times 10^5 * 1.0 - 1.5 * 1.01 \times 10^5 * 1.0) = -8.26 \times 10^4 \text{ J}$
 - (4) $\Delta U = \frac{3}{2}(1.0 * 1.01 \times 10^5 * 0.5 - 1.0 * 1.01 \times 10^5 * 1.0) = -8.26 \times 10^4 \text{ J}$
- (d) $\Delta U_{total} = 0$ (take care of rounding errors!)
- (e) $\Delta W = - \int P dV$
- (1) $dV = 0 \rightarrow \Delta W = 0$
 - (2) $P = \text{cont}$, so $\Delta W = -P\Delta V = -1.5 * 1.01 \times 10^5 * (1.0 - 0.5) = -7.57 \cdot 10^4 \text{ J}$
 - (3) $dV = 0 \rightarrow \Delta W = 0$
 - (4) $\Delta W = -P\Delta V = -1.0 * 1.01 \times 10^5 * (0.5 - 1.0) = 5.05 \cdot 10^4 \text{ J}$
- (f) So ΔW by gas $= -7.57 \cdot 10^4 + 5.05 \cdot 10^4 = 2.52 \cdot 10^4 \text{ J}$
- (g) In each stage heat enters or leaves the system (as well as the work done). During part (1) and (2) heat must have been put into the system.
- (h) The work done is the area bounded by the path in the PV diagram ($= 0.5 * 0.5 * 1.10 \times 10^5$).

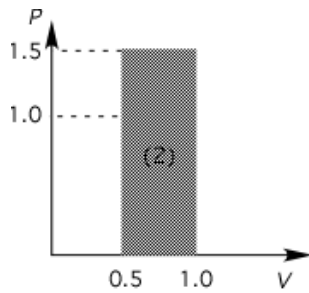


Figure 2: Work done by gas in (2)

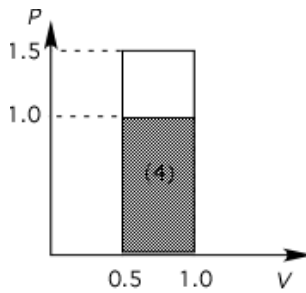


Figure 3: Work done on gas in (4)

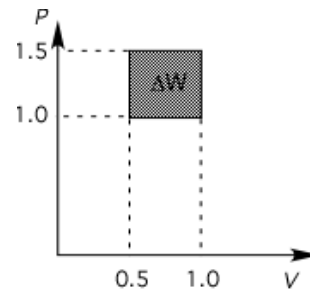


Figure 4: Total work done by gas

4. (a) $N = \frac{PV}{k_B T} = \frac{10^5 \times 1}{1.38 \times 10^{-23} \cdot 300} = 2.41 \times 10^{25}$ molecules.
 $n = N/N_A = 2.41 \times 10^{25} / 6.02 \times 10^{23} = 40.1$ moles
 $m = n * A$ (atomic mass) g $= 40.1 * 40.0 = 1.6$ kg
- (b) $C_V = \frac{3}{2}Nk_B = \frac{3}{2}nR = \frac{3}{2} * 40.1 * 8.31 = 500 \text{ JK}^{-1}$
 $C_P = \frac{5}{2} * 40.1 * 8.31 = 833 \text{ JK}^{-1}$
 $C_P/C_V = 5/3 = 1.67$.
- (c) $c_V = C_V/m = 500/1.6 = 312 \text{ JK}^{-1}\text{kg}^{-1}$
 $c_P = C_P/m = 833/1.6 = 521 \text{ JK}^{-1}\text{kg}^{-1}$
 $c_P/c_V = 1.67$.
- (d) $C_{V_m} = C_V/n = 500/40.1 = 12.5 \text{ JK}^{-1}\text{mol}^{-1}$
 $c_{P_m} = C_P/n = 833/40.1 = 20.8 \text{ JK}^{-1}\text{mol}^{-1}$
 $C_{P_m}/C_{V_m} = 1.67$.

5. (a) Blocks end up at same temp T and have same change in heat, so
 $C(T - 323) * 5 + C(T - 273) * 10 = 0 \rightarrow 15T = (273 * 10) + (323 * 5)$
 $\rightarrow T = 289.7\text{K} = 16.7^\circ\text{C}$
- (b) Mass exhaled each breath = $0.5 * 1.3 \times 10^{-3} = 6.5 \times 10^{-4} \text{ kg}$
Energy lost per breath = $1020 * 6.5 \times 10^{-4} * (37 - 20) = 11.3 \text{ J}$
Total per hour = $11.3 * 20 * 60 = 13.6 \text{ kJ}$ (equivalent to 3.8 W)
- (c) Heat flow due to radiation = $\sigma(T_1^4 - T_2^4) * A = 5.67 \times 10^{-8} (308^4 - 293^4) * 5 \times 10^{-2} = 4.6 \text{ W}$
So heat loss in one hour = $4.6 * 3600 = 16.6 \text{ kJ}$
- (d) Total energy change = $50 * 100 * 3600 = 1.8 \times 10^7 \text{ J}$
So $\Delta T = \Delta Q / C = \frac{1.8 \times 10^7}{1020 * 1.3 \times 10^{-3} * 4 \times 10^6} = 3.4^\circ\text{C}$
6. (a) $dQ = 0$ so first law becomes $dU = -PdV = -\frac{Nk_B T}{V} dV$ (using ideal gas EOS)
also $U = \frac{n_d}{2} Nk_B T \rightarrow dU = \frac{n_d}{2} Nk_B dT$
equate the above for $dU \rightarrow -\frac{Nk_B T}{V} dV = \frac{n_d}{2} Nk_B dT$
divide through to get: $\frac{dT}{T} = -\frac{2}{n_d} \frac{dV}{V}$.
- (b) $\ln T = -\frac{2}{n_d} \ln V + c_1 = \ln V^{-2/n_d} + \ln c_2$ (c_1 and c_2 are constants)
 $\rightarrow TV^{2/n_d} = c_2$.
- (c) Using $c_3 * T = PV \rightarrow P * V * V^{2/n_d} = c_2/c_3 = c_4$ (c_3 and c_4 are constants)
 $\rightarrow PV^{(2+n_d)/n_d} = c_4$
but $\gamma = \frac{C_P}{C_V} = \frac{(\frac{n_d}{2} Nk_B + Nk_B)}{\frac{n_d}{2} Nk_B} = \frac{n_d + 2}{n_d}$
So $PV^\gamma = c_4$ For ideal gas $n_d = 3$, so $\gamma = 5/3$
- (d) $P_1 V_1^\gamma = P_0 V_0^\gamma \rightarrow P_1 = P_0 (V_0/2V_0)^\gamma = P_0 (1/2)^{5/3} \approx 0.315 P_0$
For isotherm, $P_1 V_1 = P_0 V_0 \rightarrow P_1 = P_0 (V_0/2V_0) = 0.5 P_0$
So adiabat is steeper than the isotherm.

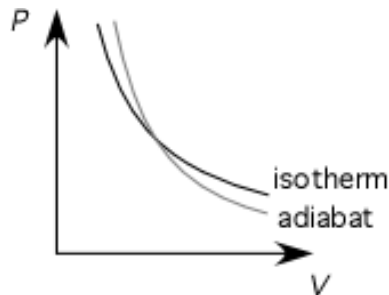


Figure 5: Isotherm and adiabat

7. (a) $dQ = 0 \rightarrow dW = dU = C_V dT$ (by definition of C_V)

$$\rightarrow \Delta W = \int W = \int U = C_V(T_1 - T_0)$$

(b) $\Delta W = - \int P dV = - \int AV^{-\gamma} dV = -\frac{1}{1-\gamma} [AV^{1-\gamma}]_{V_0}^{V_1}$

where A is a constant, but $AV^{-\gamma} = P$, so

$$\Delta W = \frac{1}{(\gamma - 1)} [P_i V_i]_0^1 = \frac{1}{(\gamma - 1)} (P_1 V_1 - P_0 V_0)$$

(c) $\gamma - 1 = (n_d + 2)/n_d - 1 = 2/n_d$, so

$$\Delta W = \frac{1}{(\gamma - 1)} (P_1 V_1 - P_0 V_0) = \frac{Nk_B}{2/n_d} (T_1 - T_0) = (n_d Nk_B/2) (T_1 - T_0)$$

$$\rightarrow \Delta W = C_V(T_1 - T_0) \text{ as before}$$

8. (a) $v_s = \sqrt{\frac{\gamma P}{\rho}}$ rearranging gives $\rightarrow \gamma = \frac{\rho v_s^2}{P} = \frac{nmv_s^2}{nk_B T} = \frac{n_d + 2}{n_d}$

$$n_d + 2 = n_d \frac{mv_s^2}{k_B T} \rightarrow n_d \left(\frac{mv_s^2}{k_B T} - 1 \right) = 2$$

$$\rightarrow n_d = 2 \left(\frac{mv_s^2}{k_B T} - 1 \right)^{-1}$$

(b) $n_d = 2 \left(\frac{4.82 \times 10^{-26} \cdot 344^2}{1.38 \cdot 10^{-23} 293} - 1 \right)^{-1} \approx 5$

(c) air is mostly diatomic, so might expect $n_d = 7$, but reason is explained more fully in lec. 7

Structure of Matter

Problem Sheet 2 Answers

1. Total probability = 1, so $\int_{-\infty}^{\infty} f(v_x) dv_x = 1$

$$\rightarrow \int_{-\infty}^{\infty} A e^{-\alpha v_x^2} dv_x = A \int_{-\infty}^{\infty} e^{-\alpha v_x^2} dv_x = A \left(\frac{\pi}{\alpha}\right)^{1/2} = 1$$

$$\rightarrow A = \left(\frac{\alpha}{\pi}\right)^{1/2} = \left(\frac{m}{2\pi k_B T}\right)^{1/2}$$

2. a) Most probable speed is a turning point (maximum) of the function $f(v) = A^3 4\pi v^2 e^{-\alpha v^2}$

(A and α have the same meanings as in Q1 here and in the following questions)

i.e. $\frac{df(v)}{dv} = 4\pi A^3 (2v e^{-\alpha v^2} - 2\alpha v^3 e^{-\alpha v^2}) = 0$

$$\rightarrow v_{mp} e^{-\alpha v^2} (1 - \alpha v^2) = 0$$

which gives the only non-trivial soln. as $v_{mp} = \left(\frac{1}{\alpha}\right)^{1/2} = \left(\frac{2k_B T}{m}\right)^{1/2}$

b) $f(v_{mp}) = 4\pi A^3 v_{mp}^2 e^{-\alpha v_{mp}^2} = 4\pi A^3 \frac{1}{\alpha} e^{-\alpha \frac{1}{\alpha}} = 4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{1}{\alpha} e^{-1}$

$$f(v_{mp}) = \left(\frac{4}{e}\right) \left(\frac{\alpha}{\pi}\right)^{1/2} = \left(\frac{4}{e}\right) \left(\frac{m}{2\pi k_B T}\right)^{1/2}$$

$$\rightarrow f(v_{mp}) \propto T^{-1/2}$$

3. $\langle v \rangle = 4\pi A^3 \int_0^{\infty} v^3 e^{-\alpha v^2} dv = 4\pi A^3 \int_0^{\infty} v^2 \cdot v e^{-\alpha v^2} dv$

(Since $\int v e^{-\alpha v^2} dv = \frac{1}{-2\alpha} e^{-\alpha v^2}$, we integrate this bit, and differentiate the v^2)

$$= 4\pi A^3 \left(\left[\frac{1}{-2\alpha} v^2 e^{-\alpha v^2} \right]_0^{\infty} + \frac{1}{2\alpha} \int_0^{\infty} 2v e^{-\alpha v^2} dv \right)$$

$$= 4\pi A^3 \left(0 + \frac{1}{\alpha} \left[\frac{1}{-2\alpha} e^{-\alpha v^2} \right]_0^{\infty} \right)$$

$$= 4\pi A^3 \frac{1}{2\alpha^2} = 4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{1}{2\alpha^2} = 2 \left(\frac{1}{\pi\alpha}\right)^{1/2}$$

using $A = \left(\frac{\alpha}{\pi}\right)^{1/2}$

$$\rightarrow \langle v \rangle = \left(\frac{8k_B T}{\pi m}\right)^{1/2}$$

4. $\langle v^2 \rangle = 4\pi A^3 \int_0^{\infty} v^4 e^{-\alpha v^2} dv = 4\pi A^3 \times \frac{3}{8} \left(\frac{\pi}{\alpha^5}\right)^{1/2}$

again making use of a standard integrals. With $A = \left(\frac{\alpha}{\pi}\right)^{1/2}$,

$$\langle v^2 \rangle = 4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{3}{8} \left(\frac{\pi}{\alpha^5}\right)^{1/2} = \frac{3}{2} \left(\frac{1}{\alpha}\right) = \left(\frac{3k_B T}{m}\right)$$

The mean kinetic energy is then $\bar{u} = \langle \frac{1}{2} m v^2 \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$

$$5. v_{mp} = \left(\frac{2k_B T}{m}\right)^{1/2} = \sqrt{2} \left(\frac{1.38 \times 10^{-23} \cdot 293}{32 \cdot 1.66 \times 10^{-27}}\right)^{1/2} = 390 \text{ ms}^{-1}$$

$$\langle v \rangle = \left(\frac{8k_B T}{\pi m}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38 \times 10^{-23} \cdot 293}{32 \cdot 1.66 \times 10^{-27}}\right)^{1/2} = 440 \text{ ms}^{-1}$$

$$\langle v^2 \rangle^{1/2} = \left(\frac{3k_B T}{m}\right)^{1/2} = \sqrt{3} \left(\frac{1.38 \times 10^{-23} \cdot 293}{32 \cdot 1.66 \times 10^{-27}}\right)^{1/2} = 478 \text{ ms}^{-1}$$

$$6. \text{ (a) } \langle v_D \rangle = \left(\frac{8k_B T}{\pi m_D}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38 \times 10^{-23} \cdot 10^8}{3.34 \times 10^{-27}}\right)^{1/2} = 1.03 \times 10^6 \text{ ms}^{-1}$$

$$\langle v_T \rangle = \left(\frac{8k_B T}{\pi m_T}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38 \times 10^{-23} \cdot 10^8}{5.01 \times 10^{-27}}\right)^{1/2} = 8.37 \times 10^5 \text{ ms}^{-1}$$

$$\langle v_e \rangle = \left(\frac{8k_B T}{\pi m_e}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38 \times 10^{-23} \cdot 10^8}{9.11 \times 10^{-31}}\right)^{1/2} = 6.21 \times 10^7 \text{ ms}^{-1}$$

(b) for all species, $\bar{u} = \frac{3}{2} k_B T$ (independent of m)

$$\bar{u} = \frac{3}{2} k_B T = \frac{3}{2} \cdot 1.38 \times 10^{-23} \cdot 10^8 = 2.07 \times 10^{-15} \text{ J} = 13,000 \text{ eV} = 13 \text{ keV}$$

$$7. \langle v_x \rangle = \int_{-\infty}^{\infty} v_x f(v_x) dv_x = \int_{-\infty}^{\infty} v_x A e^{-\alpha v_x^2} dv_x \text{ (directly integratable)}$$

$$= \lim_{u \rightarrow \infty} \left[\frac{A}{2} e^{-\alpha v_x^2} \right]_{-u}^{+u} = \frac{A}{2} \lim_{u \rightarrow \infty} [e^{-u} - e^{-u}] = 0$$

$$8. \langle z \rangle = \int_0^{\infty} z p(z) dz$$

$$= \int_0^{\infty} \frac{z}{\lambda} e^{-z/\lambda} dz = \left([-z e^{-z/\lambda}]_0^{\infty} + \int_0^{\infty} e^{-z/\lambda} dz \right) \text{ (integrating by parts)}$$

$$= 0 + [-\lambda e^{-z/\lambda}]_0^{\infty}$$

$$= 0 - \lambda(0 - 1) = \lambda$$

$$\text{So } \langle \text{p.e.} \rangle = \langle mgz \rangle = mg \langle z \rangle = mg\lambda = mg \frac{k_B T}{mg} = k_B T$$

Structure of Matter

Problem Sheet 3 Answers

$$\begin{aligned} 1. \langle v_x^2 \rangle &= \int_{-\infty}^{\infty} v_x^2 f(v_x) dv_x = 1 \\ &= \int_{-\infty}^{\infty} A v_x^2 e^{-\alpha v_x^2} dv_x = A \int_{-\infty}^{\infty} v_x^2 e^{-\alpha v_x^2} dv_x \\ &= A \frac{1}{2} \left(\frac{\pi}{\alpha^3} \right)^{1/2} = \left(\frac{\alpha}{\pi} \right)^{1/2} \frac{1}{2} \left(\frac{\pi}{\alpha^3} \right)^{1/2} \\ &= \left(\frac{1}{2\alpha} \right) = \left(\frac{k_B T}{m} \right) \end{aligned}$$

$$\begin{aligned} 2. x_1 = x - (d/2) \quad \rightarrow \quad v_1 &= \frac{dx}{dt} - \frac{d(d/2)}{dt} = v_x - (v/2) \\ x_2 = x + (d/2) \quad \rightarrow \quad v_2 &= \frac{dx}{dt} + \frac{d(d/2)}{dt} = v_x + (v/2) \end{aligned}$$

$$\text{Total ke} = \frac{1}{2}(m/2)v_1^2 + \frac{1}{2}(m/2)v_2^2 \quad (\text{each particle has mass } m/2)$$

$$\begin{aligned} &= \frac{m}{4}(v_x - (v/2))^2 + \frac{m}{4}(v_x + (v/2))^2 = 2 \times \frac{m}{4}v_x^2 + 2 \times \frac{m}{4}(v/2)^2 \\ &= \frac{1}{2}mv_x^2 + \frac{1}{2} \left(\frac{m}{4} \right) v^2 \end{aligned}$$

$$3. \quad (\text{a}) \quad U(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

$$\rightarrow \frac{dU}{dr} = -12 \frac{A}{r^{13}} + 6 \frac{B}{r^7} = 0$$

at equilibrium.

$$\rightarrow -12 \frac{A}{r_0^{13}} + 6 \frac{B}{r_0^7} = 0$$

$$\rightarrow 12A = 6Br_0^6$$

$$\rightarrow r_0 = (2A/B)^{1/6}$$

$$(\text{b}) \quad U(r_0) = \frac{A}{r_0^{12}} - \frac{B}{r_0^6} = \frac{AB^2}{4A^2} - \frac{BB}{2A} = -\frac{1}{4} \frac{B^2}{A}$$

$$(\text{c}) \quad -\epsilon = -\frac{1}{4} \frac{B^2}{A} \quad \rightarrow \quad (1) \quad \epsilon = \frac{1}{4} \frac{B^2}{A}$$

(using result of part (b))

$$\text{AND} \quad (2) \quad r_0^6 = \frac{2A}{B}$$

multiply together (1) and (2), $2\epsilon r_0^6 = B$

and $(1) \times (2)^2$, $2\epsilon r_0^{12} = A$

So substituting for A and B in the Lennard-Jones expression,

$$U(r) = \epsilon \left\{ \left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right\}$$

$$(d) \frac{d^2U}{dr^2} = \left\{ (12 \times 13 \epsilon) \frac{r_0^{12}}{r^{14}} - (12 \times 7 \epsilon) \frac{r_0^6}{r^8} \right\} \times 2$$

(the 2 is because each atom is bounded on either side, so has twice the potential energy)

$$\text{For } r = r_0, \quad \frac{d^2U}{dr^2} = (12 \times 6 \epsilon) \frac{1}{r_0^2} \times 2 = \frac{144\epsilon}{r_0^2}$$

$$\text{and so } \omega_E = \sqrt{\frac{144\epsilon}{mr_0^2}}$$

$$\text{and } v_s = \omega\lambda/2\pi = (r_0/2\pi) \sqrt{\frac{144\epsilon}{mr_0^2}} = \frac{1}{2\pi} \sqrt{\frac{144\epsilon}{m}}$$

$$4. (a) U = \frac{A}{x^{12}} - \frac{B}{x^n}$$

$$\rightarrow \frac{dU}{dx} = -12 \frac{A}{x^{13}} + n \frac{B}{x^{n+1}} = 0$$

at equilibrium

also at equilibrium $x = 1$, so $-12A + nB = 0$

$$\rightarrow B = 12A/n$$

$$\text{So } U(x = 1) = -\epsilon = A - B = A - 12A/n$$

$$\rightarrow \epsilon = A(12/n - 1)$$

$$(b) \text{ For } n = 1 \quad \rightarrow \quad \epsilon = A \cdot 11,$$

$$\text{for } n = 6 \quad \rightarrow \quad \epsilon = A,$$

Assuming repulsive term is the same in both cases (i.e. value of A is similar), this suggests that the binding potential for the ionic bonding is $11 \times$ stronger.

5. (a) Collision just happens if centre of particles is within $2a$ (the radius of both combined) of the other.

$$\sigma = 4\pi a^2$$

(NB this is different from the expression given in the lectures because a is the radius here, whereas the diameter d was used in lec. 5.)

(b) Volume swept out by this cross-section is $4\pi a^2 l$

Number of particles in this volume = $4\pi a^2 l n$
(n is number density)

If particles is assumed to not be deflected significantly from its path, this is the number of collisions in this path length too.

(c) So average distance between collisions $\lambda_m = l/4\pi a^2 l n = 1/4\pi a^2 n$

(d) Using $P = nk_B T \rightarrow n = P/k_B T$ and including the extra $\sqrt{2}$

$$\rightarrow \lambda_m = k_B T / 4\sqrt{2}\pi a^2 P$$

$$(e) \lambda_m = \frac{1.38 \times 10^{-23} \cdot 293}{4\sqrt{2}\pi (2 \times 10^{-10})^2 \cdot 1.01 \times 10^5} = 5.63 \times 10^{-8} \text{ m} = 56 \text{ nm}$$

$$(f) \text{ mean time between collisions } \tau_m = \frac{\lambda_m}{\bar{v}} = \frac{5.63 \times 10^{-8}}{440} = 1.28 \times 10^{-10} \text{ s} = 128 \text{ ps}$$

6. For monoatomic van der Waal gas, $U = \frac{3}{2}Nk_B T - aN^2/V$

$$\text{but } C_V = \left(\frac{dU}{dT}\right)_V = \frac{3}{2}Nk_B$$

the same as for an ideal monoatomic gas

7. Volume available to a real gas = $(V - bN)$,

fractional change = bN and % change from ideal gas = bN/V

but $b \cong 4V_m$ where V_m is the effective collisional volume of the molecule, i.e. $V_m = \frac{4}{3}\pi a^3$

using the ideal gas eqn. fractional change $a \cong b n = \frac{16}{3}\pi a^3 \frac{P}{k_B T}$

$$\rightarrow P = \frac{a \cdot 3k_B T}{16\pi a^3} = \frac{0.01 \times 3 \times 1.38 \times 10^{-23} \cdot 293}{16\pi(2 \times 10^{-10})^3} = 3.02 \times 10^5 \text{ Pa} \approx 3 \text{ atm}$$

8. van der Waal's equation: $P = \frac{Nk_B T}{(V - bN)} - \frac{aN^2}{V^2}$

$$\frac{dP}{dV} = -\frac{Nk_B T}{(V - bN)^2} + \frac{2aN^2}{V^3} = 0 \quad \rightarrow \quad \frac{Nk_B T}{(V - bN)^2} = \frac{2aN^2}{V^3}$$

$$\frac{d^2P}{dV^2} = \frac{2Nk_B T}{(V - bN)^3} - \frac{6aN^2}{V^4} = 0 \quad \rightarrow \quad \frac{2Nk_B T}{(V - bN)^3} = \frac{6aN^2}{V^4}$$

dividing one expression by the other $(V - bN)/2 = 2V/6$

$$3(V - bN) = 2V \quad \rightarrow \quad V_c = 3bN$$

Put this back into the first derivative: $Nk_B T = \frac{2aN^2}{V^3}(V - bN)^2$

$$T = \frac{2aN}{k_B(3bN)^3} (2bN)^2$$

$$\rightarrow T_c = \frac{8a}{27k_B b}$$

and finally in VW equation of state;

$$P_c = \frac{Nk_B 8a}{27k_B b(2bN)} - \frac{aN^2}{(3bN)^2} = \frac{4a}{27b^2} - \frac{a}{9b^2} = \frac{a}{27b^2}$$

Structure of Matter

Problem Sheet 4 Answers

1. (a) Iceberg is in equilibrium so

$$\text{weight of berg} = \text{buoyant force on cube} = \text{weight of water displaced}$$

$$\text{If volume displaced} = V_1 \text{ and total volume} = V$$

$$\rightarrow \rho_{ice} V g = \rho_{water} V_1 g$$

$$\text{hence fraction submerged} = V_1/V = \rho_{ice}/\rho_{water} = 920/1025 = 0.898 \text{ (89.8\%)}$$

- (b) For balloon to be in equilibrium,

$$\text{weight of balloon} + \text{weight of helium filled} = \text{weight of air displaced}$$

$$5 \times 10^{-3} g + \frac{4}{3}\pi r^3 * \rho_{He} g = \frac{4}{3}\pi r^3 * \rho_{air} g$$

$$5 \times 10^{-3} = \frac{4}{3}\pi r^3 n_0 (m_{air} - m_{He})$$

$$r^3 = \frac{5 \times 10^{-3}}{\frac{4}{3}\pi (1.01 \times 10^5/k_B * 273) (29 - 4) 1.66 \times 10^{-27}}$$

$$r = 0.10 \text{ m} = 10 \text{ cm}$$

2. (a) Incompressible so $u_1 A_1 = u_2 A_2$

$$u_2 = \pi 0.8^2 * 1.2 / \pi 0.4^2 = 4.8 \text{ m/s}$$

- (b) Flow rate = $u \cdot A = \pi (0.4 \times 10^{-2})^2 * 4.8 = 2.41 \times 10^{-4} \text{ m}^3/\text{s}$

$$\text{So time taken to fill tank} = 20 / 2.41 \times 10^{-4} = 8.29 \times 10^4 \text{ s} \approx 23 \text{ hours}$$

- (c) Bernoulli's equation (ignoring gravity);

$$P_1 + \frac{1}{2}\rho u_1^2 = P_2 + \frac{1}{2}\rho u_2^2 \quad \rightarrow \quad P_2 = P_1 + \frac{1}{2}\rho(u_1^2 - u_2^2)$$

$$P_2 = 2 * (1.01 \times 10^5) + \frac{1}{2} 1000 (4.8^2 - 1.2^2) = 2.02 \times 10^5 - 1.08 \times 10^4$$

$$= 1.91 \times 10^5 \text{ Pa} = 1.89 \text{ bar}$$

3. (a) Effective volume is $b = 4V_m = 4\frac{4}{3}\pi r^3$

$$\text{so } r = (3b/16\pi)^{1/3} = (3 \cdot 6.49 \times 10^{-29}/16\pi)^{1/3} = 1.57 \times 10^{-10} \text{ m} = 1.57 \text{ \AA}$$

$$(b) T_c = \frac{8a}{27k_B b} = \frac{8 \cdot 3.86 \times 10^{-49}}{27k_B 6.49 \times 10^{-29}} = 128 \text{ K}$$

4. (a) 3 directions of vibration with 2 degrees of freedom for each direction (one pe and one ke) $\rightarrow n_d = 6$

$$U = 6 \times \frac{1}{2} N k_B T = 3 N k_B T,$$

$$C_V = \left(\frac{\partial Q}{\partial T} \right)_V = \left(\frac{dU}{dT} \right) \text{ (from first law with no work done due to constant V)}$$

$$\rightarrow C_V = 3 N k_B$$

(b) For 1 mole $C_{V_m} = 3 N_A k_B = 3R = 24.9 \text{ J/K per mole} = 5.96 \text{ cal/K}$

(c) 27 g of Al = $27/A_{Al} = 1$ mole; 63.5 g of Al = $63.5/A_{Cu} = 1$ mole

$$\Delta U_{Al} = \Delta U_{Cu} = 0$$

$$3R \cdot \chi \cdot (T_f - 283) + 3R \cdot \chi \cdot (T_f - 373) = 0$$

$$2T_f = 283 + 373 \quad \rightarrow \quad T_f = 328 \text{ K} = 55^\circ \text{C}$$

5. (a) $\Delta L = 23.1 \times 0.3 \times (77 - 293) \mu\text{m} = -1.5 \times 10^3 \mu\text{m} = -1.5 \text{ mm}$

(a) $\Delta L = 1.17 \times 10^{-5} \times 320 \times (-30) = -0.112 \text{ m} \sim -11 \text{ cm}$

(b) Volume $V = L^3 = \{L_0(1 + \alpha\Delta T)\}^3 = L_0^3(1 + \alpha\Delta T)^3$

$$\approx L_0^3(1 + 3\alpha\Delta T + \dots) \quad \text{ignoring terms in } \alpha^2 \text{ or higher due to smallness}$$

Since $V_0 = L_0^3$, then $\Delta V = V_0 3\alpha\Delta T = V_0 \beta\Delta T$, i.e. $\beta = 3\alpha$

6. (a) State zero (ground) has probability $p_0 = Ae^{-E/k_B T}$
 State one has probability $p_1 = Ae^{-(E+\Delta E)/k_B T}$

Total probability = 1,

$$\text{so } Ae^{-E/k_B T} + Ae^{-(E+\Delta E)/k_B T} = 1$$

$$\rightarrow A = 1/\{e^{-E/k_B T}(1 + e^{-\Delta E/k_B T})\}$$

$$\text{So } N_0 = Np_0 = \frac{N}{(1 + e^{-\Delta E/k_B T})}$$

$$\text{and } N_1 = Np_1 = \frac{Ne^{-\Delta E/k_B T}}{(1 + e^{-\Delta E/k_B T})}$$

(b) Relative proportions $N_1/N_0 = e^{-\Delta E/k_B T}$, so

i. $k_B T = 0.1 \Delta E$ *ratio* = $e^{-0.1} = 4.5 \times 10^{-5}$

ii. $k_B T = 0.3 \Delta E$ *ratio* = 3.6×10^{-2}

iii. $k_B T = \Delta E$ *ratio* = 0.368

iv. $k_B T = 10 \Delta E$ *ratio* = 0.905

(c) $E = 0.023 * 1.602 \times 10^{-19} = 3.68 \times 10^{-21} J = k_B T$

$T = 267 \text{ K}$. This compares with its melting point which is 312 K

(d) $E = 4.2 * 1.602 \times 10^{-19} = 6.73 \times 10^{-19} J = k_B T$

$T = 48,800 \text{ K}$. This corresponds to conversion to a plasma

(e) $\lambda = h/p > n^{-1/3}$, but $\langle p \rangle^2/2m = \frac{3}{2}k_B T$

$$\rightarrow \langle p \rangle = (3k_B T m)^{1/2}$$

$$\rightarrow h/(3k_B T m)^{1/2} > n^{-1/3}$$

$$\rightarrow T < (h^2/3mk_B) n^{2/3}$$

(f) $n = 2000/(1 \times 10^{-5})^3 = 2 \times 10^{18} \text{ m}^{-3}$

$$\text{So } T < ((6.626 \times 10^{-34})^2/(3 \cdot 86 \cdot 1.66 \times 10^{-27} \cdot 1.38 \times 10^{-23})) (2 \times 10^{18})^{2/3}$$

$$T < 1.18 \times 10^{-7} \text{ K} = 118 \text{ nK}$$

$$\text{Corresponding energy for this temperature} = k_B T = 1.18 \times 10^{-7} \cdot 1.38 \times 10^{-23} = 1.63 \times 10^{-30} \text{ J} = 10^{-11} \text{ eV}$$

7. (a) For adiabatic change $PV^\gamma = A$ (= constant)

$$P = AV^{-\gamma} \rightarrow \frac{dP}{dV} = -A\gamma V^{-\gamma-1} = -\gamma P/V$$

$$\text{So } K = \gamma P \text{ and } c_s = \sqrt{K/\rho} = \sqrt{\gamma P/\rho}$$

(b) But for ideal gas $\rho = n_0 \times m = \frac{mP}{k_B T}$

$$\text{so } c_s = \sqrt{\gamma k_B T/m}$$

$$\text{For air, } m = 29 * 1.66 \times 10^{-27}, \gamma = 1.4, c_s = \sqrt{\frac{1.4 * k_B * 273}{29 * 1.66 \times 10^{-27}}} = 331 \text{ m/s}$$

$$\text{For He, } m = 4 * 1.66 \times 10^{-27}, \gamma = 5/3, c_s = \sqrt{\frac{(5/3) * k_B * 273}{4 * 1.66 \times 10^{-27}}} = 973 \text{ m/s}$$

8. (a) The E field can be obtained from Gauss's law, consider a cylinder, with one flat face (of area A) in the plane between the separated charges, and the other far from the charges where the E field is zero. By symmetry there is no field through the curved sides, so integrating over the side which has only ions in a layer of thickness δ

$$E \cdot A = q/\epsilon_0 = n_0 \cdot e \cdot \delta \cdot A/\epsilon_0$$

$$\rightarrow E = n_0 e \delta / \epsilon_0$$

(b) $m_e \frac{d\delta}{dt} = -e E_{max} = -e n_0 e \delta / \epsilon_0$

$$\rightarrow \delta'' + \left(\frac{n_0 e^2}{m_e \epsilon_0} \right) \delta = 0$$

which is SHM with angular frequency $\omega_p = \left(\frac{n_0 e^2}{m_e \epsilon_0} \right)^{1/2}$ - the plasma frequency

(d) $u_E = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \epsilon_0 (n_0 e \delta / \epsilon_0)^2 = \frac{1}{2} (n_0^2 e^2 \delta^2 / \epsilon_0)$

(e) $\frac{1}{2} n_0 k_B T = \frac{1}{2} (n_0^2 e^2 \delta^2 / \epsilon_0)$

$$\text{rearrange to get, } \delta_{max} = \left(\frac{\epsilon_0 k_B T}{n_0 e^2} \right)^{1/2}$$

(f) $\omega_p = \left(\frac{n_0 e^2}{m_e \epsilon_0} \right)^{1/2} = \left(\frac{10^{20} e^2}{9.1 \times 10^{-31} \cdot 8.85 \times 10^{-12}} \right)^{1/2} = 5.6 \times 10^{11} \text{ Hz}$

$$\delta_{max} = \left(\frac{\epsilon_0 k_B T}{n_0 e^2} \right)^{1/2} = \left(\frac{8.85 \times 10^{-12} \cdot 1.38 \times 10^{-23} \cdot 10^8}{10^{20} \cdot (1.60 \times 10^{-19})^2} \right)^{1/2} = 6.91 \times 10^{-5} \text{ m}$$