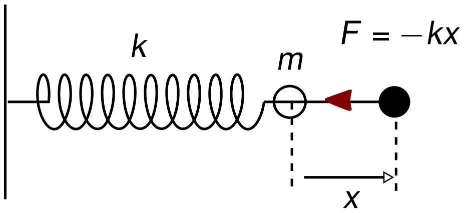
Vibrations and Waves
Formulae Summary: Lectures 1-2

Simple Harmonic Motion



Hooke's law: $F = -kx$, where k is the spring constant ($k > 0$ for restoring force).

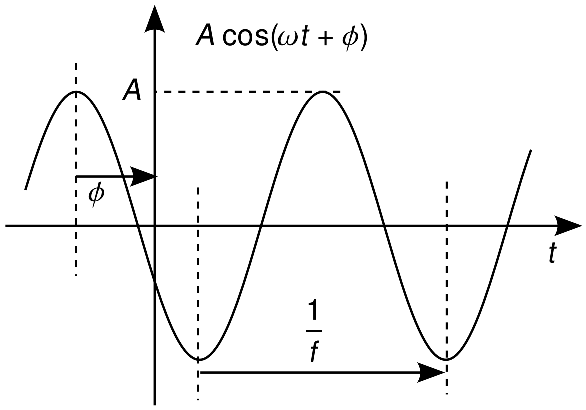
The equation of motion for a mass on a spring is

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \left(\text{notation: } \ddot{x} \equiv \frac{d^2x}{dt^2}\right)$$

which has the general solution

$$x = A\cos(\omega t + \phi)$$

($x = A\sin(\omega t + \phi)$ and $A\cos(\omega t) + B\sin(\omega t)$, are also valid as general solutions, but we use the cos form in the lectures.)



- $\omega = \sqrt{k/m}$ is the *angular frequency* which is independent of amplitude
- A is the *amplitude*
- ϕ is the *phase angle*
- $f = \omega/(2\pi)$ is the *frequency* and is measured in Hertz(Hz) (cycles per second)
- $T = 1/f = 2\pi/\omega$ is the *period*.

ω is determined by "the physics" whereas A and ϕ are adjustable parameters which we choose to fit the initial conditions. A one-dimensional simple harmonic oscillator has two degrees of freedom, corresponding to x and v,

$$x(t) = A\cos(\omega t + \phi) \quad \text{and} \quad v(t) = -\omega A\sin(\omega t + \phi)$$

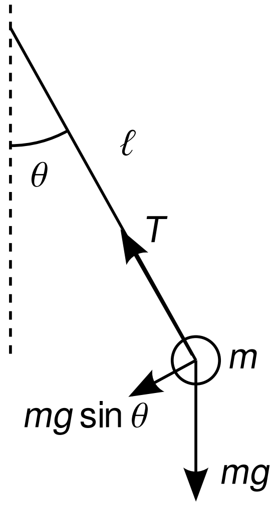
Initial conditions on $x(0)$ and $v(0)$ can then be used to determine the adjustable constants A and ϕ:

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2 \quad \text{and} \quad \tan\phi = -\frac{v(0)}{\omega x(0)}$$

[*See Handout 2 for complex notation, phasors, and velocity and acceleration. See "Extra Notes/SHM in an LC circuit" on Blackboard for the LC circuit.*]

Equivalences between the LC system and the mass-on-a-spring system:

$$x \to Q \qquad m \to L \qquad k \to \frac{1}{C} \qquad \dot{x} \to I \qquad F \to V$$



The restoring force for a pendulum is $F = -mg\sin\theta$. Applying Newton's second law and approximating $\sin\theta \approx \theta$ gives the equation of motion for SHM,

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \qquad \text{with} \quad \omega = \sqrt{\frac{g}{\ell}}$$

where ℓ is the pendulum length,

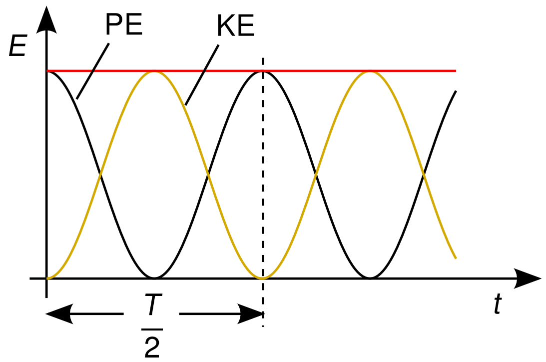
[*See Handout 3 for non-linearities and their effect on the pendulum period. See Problem Sheet 1 for linear approximations to potentials*]

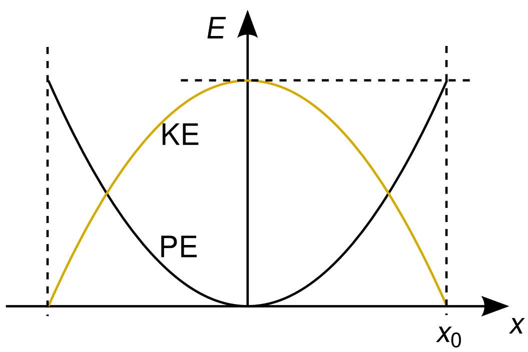
The energy for a mass on a spring is

Potential Energy: $\text{PE} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$

Kinetic Energy: $\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$

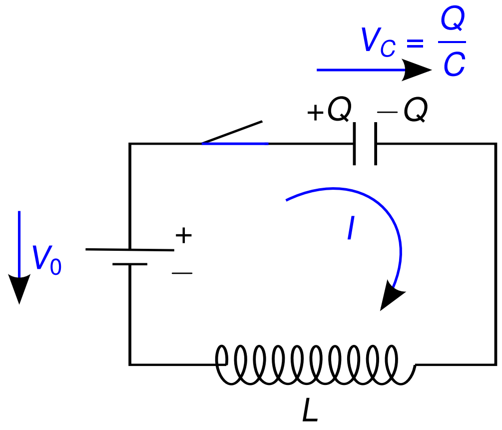
Total Energy: $E = \text{PE} + \text{KE} = \frac{1}{2}kA^2\underbrace{(\cos^2 + \sin^2)}_{=1} = \frac{1}{2}kA^2$





Vibrations and Waves - SHM in an LC circuit (Lecture 2)

A number of you asked for clarification on how we got from an equation in terms of the charge on the capacitor Q to one in terms of Q' and quite what this meant.



By considering the current and electromotive force around the circuit and applying Faraday's law (that the e.m.f around a circuit equals minus the rate of change of magnetic flux through the circuit), we obtained

$$V_C - V_0 = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

(*Please don't worry if you don't understand this bit yet!*) The voltage on the capacitor is $V_C = Q/C$. The current is the rate of change of charge on the capacitor, $I = dQ/dt$, which gives us

$$\frac{Q}{C} - V_0 = -L\frac{d^2Q}{dt^2}$$

We could have tried to solve this directly by substituting in trial solutions, but I wanted to see if we could get it into a form that we could recognise as a simple harmonic oscillator. We can do this by making the simple substitution for variables $Q' = Q - V_0C$ which gives

$$\frac{Q'}{C} = -L\frac{d^2Q'}{dt^2} + L\cancelto{0}{\frac{d^2V_0C}{dt^2}} \qquad \Rightarrow \qquad \frac{Q'}{C} = -L\frac{d^2Q'}{dt^2}$$

(where the cancellation is because V_0C is a constant, so differentiating gives zero.) Rearranging we get

$$\ddot{Q}' + \frac{1}{LC}Q' = 0 \qquad \text{— Equation of motion for SHM!}$$

Since we recognise this, we can write the solution without any further work:

$$Q' = A\cos(\omega t + \phi) \quad \text{with} \quad \omega = \sqrt{\frac{1}{LC}}$$

Substituting back to get to our original Q, (and writing the constant V_0C as $Q_0 = V_0C$) we get our solution

$$Q = Q' + Q_0 = A\cos(\omega t + \phi) + Q_0$$

You will often find that physicists take a shortcut approach of just ignoring a constant in the differential equation, then just adding it back into the final solution (it sounds dirty, but it amounts to the same thing as the formal substitution really and is less effort.)

Finally, applying our initial conditions $Q = 0$ and $\mathrm{d}Q/\,\mathrm{d}t = 0$ at $t = 0$ (as we did in lecture 1) fixes the adjustable parameters to $A = Q_0$ and $\phi = \pi$, giving,

$$Q = Q_0 \cos(\omega t + \pi) + Q_0$$

I don't expect you to be able to use Faraday's law yet, since you've not been taught it formally, nor do I expect you to be comfortable with electrical circuits. But I do expect you to understand why we manipulated the resulting equation to get it into the form that we did and that having done so we could identify it as the equation of motion for a simple harmonic oscillator and hence write down the solution for how it will behave. If YOU can do that for a system that maybe you've not seen before you'll be able to predict how it behaves without having to understand all the underlying details — that's a useful skill!

Vibrations and Waves
Formulae Summary: Lectures 3-4

Damped Oscillations

The damping force (in fluid) is:

$$\mathbf{F} = -b_1\mathbf{v} - b_2|v|^2\hat{\mathbf{v}}$$

where the first (linear) term is the viscous damping and the second (quadratic) term is the pressure term. If $b_2v^2 \ll b_1v$ then we have linear damping, which is what we will use, but be aware that there are situations where the pressure term dominates (such as lecture theatre-sized demonstrations of pendulums!)

The equation of motion for a linear damped mass on a spring is

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\text{or} \qquad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \qquad \text{where} \qquad \omega_0^2 = \frac{k}{m} \quad \gamma = \frac{b}{m}$$

Henceforth we will use ω_0 for the *natural angular frequency* of the undamped oscillator.

Units: $[\gamma] = [\omega_0] = \text{seconds}^{-1}$

We **define** a Q-factor, $Q = \dfrac{\omega_0}{\gamma}$.

We can solve the free damped oscillator equation in the complex plane by substituting in the trial solution $\tilde{x} = \tilde{A}\,\mathrm{e}^{\alpha t}$ (where $\tilde{A} = A\,\mathrm{e}^{\mathrm{i}\phi}$). This gives

$$[\alpha^2 + \alpha\gamma + \omega_0^2]\,\tilde{A}\,\mathrm{e}^{\alpha t} = 0$$

which has solutions for α given by

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

Light damping ($\frac{\gamma^2}{4} < \omega_0^2$)

$$\alpha = -\frac{\gamma}{2} \pm \mathrm{i}\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

which gives an oscillatory solution with an exponentially decaying amplitude:

$$\tilde{x} = \tilde{A}\,\mathrm{e}^{-\frac{\gamma}{2}t}\,\mathrm{e}^{\pm\mathrm{i}\omega_\mathrm{d} t} \qquad \text{where,} \qquad \omega_\mathrm{d}^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

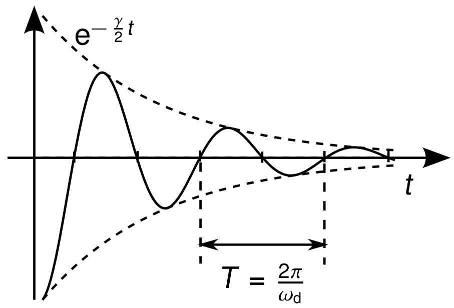
$$\text{Real part:} \qquad x = A\,\mathrm{e}^{-\frac{\gamma}{2}t}\cos(\omega_\mathrm{d} t + \phi)$$

- ω_d is the *damped frequency* which is independent of amplitude.
- Adjustable constants A and ϕ will be determined by initial conditions.

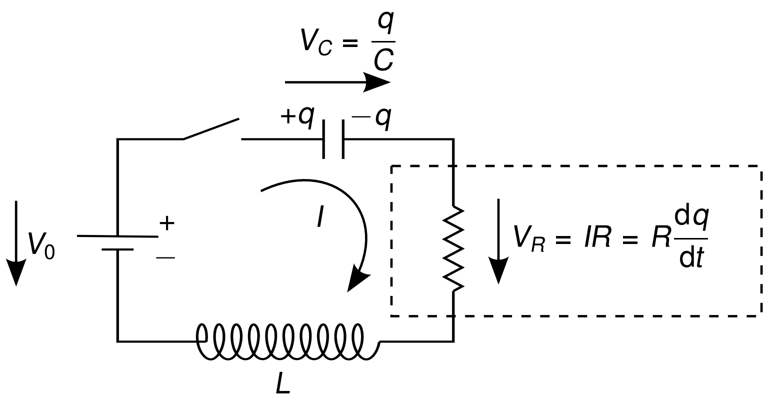
In terms of Q,

$$\omega_\mathrm{d} = \omega_0\sqrt{1 - \frac{1}{4Q^2}} = \omega_0\left[1 - \frac{1}{8Q^2} + \ldots\right]$$

from which we see that the difference between ω_d and ω_0 decreases rapidly as Q increases.



- The amplitude ($A\,\mathrm{e}^{-\frac{\gamma}{2}t}$) decays by $1/\mathrm{e}$ in a time $2/\gamma$.
- The energy, which is proportional to the amplitude squared decays by $1/\mathrm{e}$ in a time $1/\gamma$.
- $Q = \omega_0/\gamma = \pi N$ where N is the number of oscillations for the amplitude to decay by $1/\mathrm{e}$.



For an LCR circuit,

$$V_\mathrm{C} + V_R - V_0 = -L\frac{\mathrm{d}I}{\mathrm{d}t} \qquad \Rightarrow \qquad \frac{q}{C} + R\frac{\mathrm{d}q}{\mathrm{d}t} - V_0 = -L\frac{\mathrm{d}^2q}{\mathrm{d}t^2}$$

This is the same as the LC circuit in Lecture 2 with the addition of the R term. Rearranging (and dropping the constant V_0 which will just add a constant to the solution), this reduces to the standard form

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0 \qquad \text{where} \qquad \omega_0 = \sqrt{\frac{1}{LC}} \qquad \gamma = \frac{R}{L}$$

We can add $b \to R$ to the list of equivalences we identified for the LC circuit in Lecture 2.

From Lecture 2, the energy in a harmonic oscillator is $E = \frac{1}{2}kA^2$. Applying this to the damped oscillator gives

$$E(t) = \frac{1}{2}kA_0^2\,\mathrm{e}^{-\gamma t}$$

For high Q,

$$\frac{Q}{2\pi} \approx \frac{\text{Energy stored in oscillator}}{\text{Energy lost per period}}$$

Heavy Damping ($\frac{\gamma^2}{4} > \omega_0^2$)

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

In this case α is real and there is no oscillatory motion.

$$x = \mathrm{e}^{-\frac{\gamma}{2}t} \left[A_1 \, \mathrm{e}^{-t\sqrt{\frac{\gamma^2}{4}-\omega_0^2}} + A_2 \, \mathrm{e}^{+t\sqrt{\frac{\gamma^2}{4}-\omega_0^2}} \right]$$

We can also rewrite the solution of the form

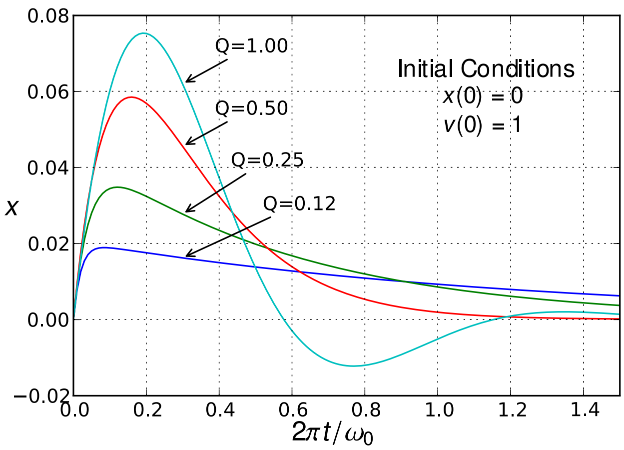
$$x = \mathrm{e}^{-\frac{\gamma}{2}t} \left[B_1 \cosh\left(t\sqrt{\frac{\gamma^2}{4} - \omega_0^2} \right) + B_2 \sinh\left(t\sqrt{\frac{\gamma^2}{4} - \omega_0^2} \right) \right]$$

This form is simpler if the initial conditions are of the form $x = 0$ $v = v_0$, (corresponding to the oscillator receiving an impulse), since the cosh term must then be zero.

Critical Damping ($\frac{\gamma^2}{4} = \omega_0$)

In this case, our previous trial solution leads to a solution with only one adjustable parameter, and so it is not the general solution. Without proof, here is the general solution for the critically damped oscillator

$$x = (A + Bt) \, \mathrm{e}^{-\frac{\gamma}{2}t}$$



Plots of displacement against time for oscillators with different Q with initial conditions corresponding to an impulse. The critically damped oscillator decays to zero fastest.

$Q > 0.5$	light damping
$Q = 0.5$	critical damping
$Q < 0.5$	heavy damping

Linear superposition and beating

Consider the superposition of two oscillations of similar frequencies ω_1 and ω_2 of equal amplitude (such as the superposition of sound oscillations from two wine glasses for instance)

$$\tilde{x} = A \, \mathrm{e}^{\mathrm{i}\omega_1 t} + A \, \mathrm{e}^{\mathrm{i}\omega_2 t}$$

Rewrite ω_1 and ω_2,

$$\omega_1 = \frac{\omega_1 + \omega_2}{2} + \frac{\omega_1 - \omega_2}{2} \qquad \text{and} \qquad \omega_2 = \frac{\omega_1 + \omega_2}{2} - \frac{\omega_1 - \omega_2}{2}$$

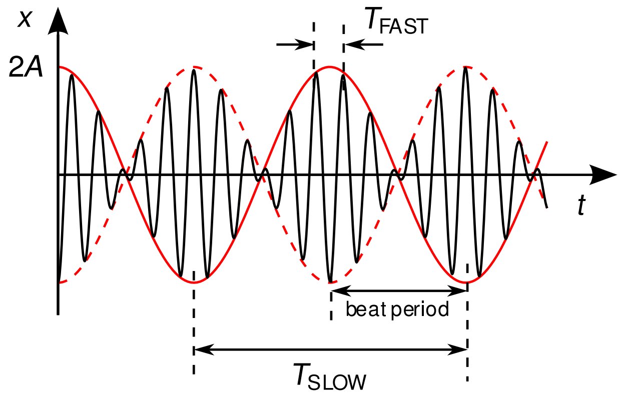
then

$$\tilde{x} = A \, \mathrm{e}^{\mathrm{i}\frac{\omega_1+\omega_2}{2}t} \underbrace{\left[\mathrm{e}^{\mathrm{i}\frac{\omega_1-\omega_2}{2}t} + \mathrm{e}^{-\mathrm{i}\frac{\omega_1-\omega_2}{2}t} \right]}_{2\cos\left(\frac{\omega_1-\omega_2}{2}t\right)}$$

Taking the real part,

$$x = 2A\cos\underbrace{\left(\frac{\omega_1+\omega_2}{2}t\right)}_{\text{FAST}}\cos\underbrace{\left(\frac{\omega_1-\omega_2}{2}t\right)}_{\text{SLOW}}$$

$$x = \mathrm{Re}[\tilde{x}] = 2A\cos\underbrace{(\bar{\omega}t)}_{\text{FAST}}\cos\underbrace{\left(\tfrac{\Delta\omega}{2}t\right)}_{\text{SLOW}} \qquad \bar{\omega} = \frac{\omega_1+\omega_2}{2} \qquad \Delta\omega = \omega_1-\omega_2$$



$$\frac{T_{\text{SLOW}}}{T_{\text{FAST}}} = \frac{\omega_1+\omega_2}{|\omega_1-\omega_2|}$$

[In the lecture we verified this for beating between oscillations from two different glasses by counting the number oscillations in the period T_{SLOW}.]

We hear modulation at the *beat frequency*,

$$\Delta f = |f_1 - f_2| = \frac{2}{T_{\text{SLOW}}}$$

(Remember: T_{SLOW} is not the beat period!)

Vibrations and Waves
Formulae Summary : Lectures 5–6

Forced Oscillations

The equation of motion for a damped oscillator driven by a force $F_0 \cos \omega t$ is

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

$$\text{or} \quad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t \quad \text{where} \quad \gamma = \frac{b}{m}, \quad \omega_0^2 = \frac{k}{m}$$

$$\text{or} \quad \ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2 \tilde{x} = \frac{F_0}{m}\mathrm{e}^{\mathrm{i}\omega t} \quad \text{(complex plane)}$$

We solve by substituting in the trial solution $\tilde{x} = A\mathrm{e}^{\mathrm{i}\omega t+\phi}$. This determines the amplitude A and phase ϕ of the response,

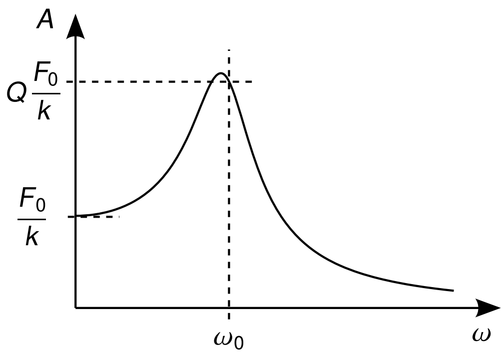
$$A\mathrm{e}^{\mathrm{i}\phi} = \frac{F_0/m}{(\omega_0^2 - \omega^2) + \mathrm{i}(\omega\gamma)} \quad \Rightarrow \quad A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \qquad \tan\phi = \frac{-\omega\gamma}{\omega_0^2 - \omega^2}$$

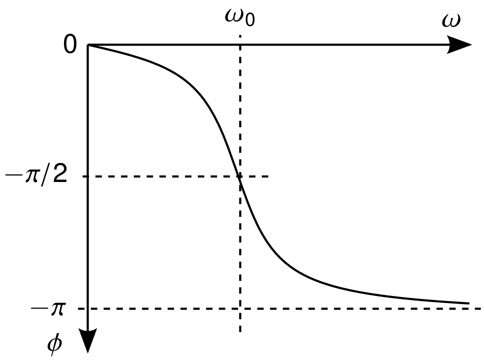
$A\mathrm{e}^{\mathrm{i}\phi}$ describes the response of the system to the driving force at angular frequency ω. Note that there are no adjustable parameters in the steady state solution.

$$\omega \ll \omega_0 : \quad A \approx \frac{F_0}{k}, \quad \phi \approx 0 \quad \text{—“stiffness controlled”}$$

$$\omega \gg \omega_0 : \quad A \approx \frac{F_0/m}{\omega^2} \quad \phi \approx -\pi \quad \text{—“mass controlled”}$$

$$\omega = \omega_0 : \quad A = \frac{F_0/m}{\omega_0\gamma} = \frac{F_0}{k}Q \quad \phi = -\frac{\pi}{2} \quad \text{—“resistance limited”}$$





The maximum amplitude response occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \quad \text{at which} \quad A_{\text{res}} = A_{\omega_0}\frac{1}{\sqrt{1 - 1/(4Q^2)}}$$

The difference between this and the value at ω_0 is very small for high Q.

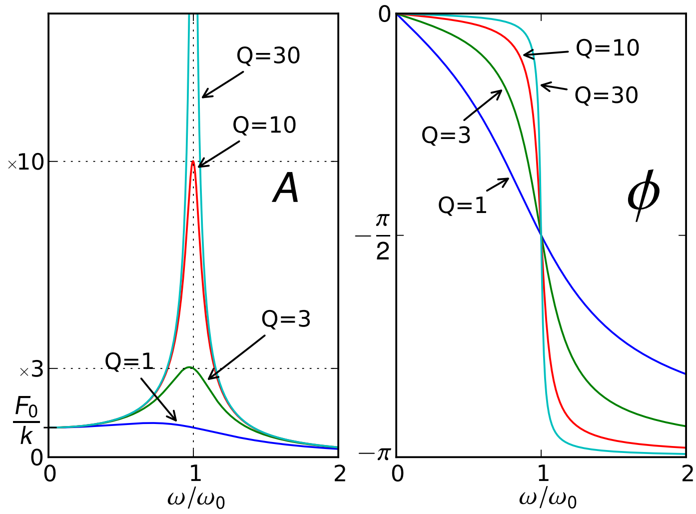
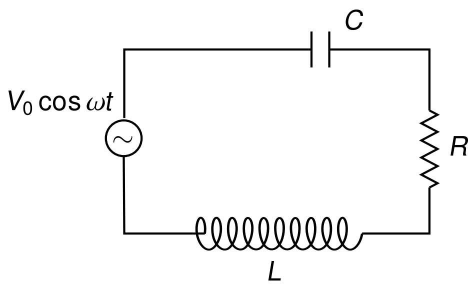


Figure 1: Forced damped oscillator response



For an LCR circuit driven by a cosinusoidal voltage, $V_0 \cos \omega t$ the equation of motion (for the capacitor charge) is

$$\ddot{q} + \gamma \dot{q} + \omega_0^2 q = \frac{V_0}{L} \cos \omega t \qquad \text{where} \qquad \omega_0 = \sqrt{\frac{1}{LC}} \qquad \gamma = \frac{R}{L}$$

and we can add $F_0 \to V_0$ to the list of equivalences between the mechanical and electrical systems.

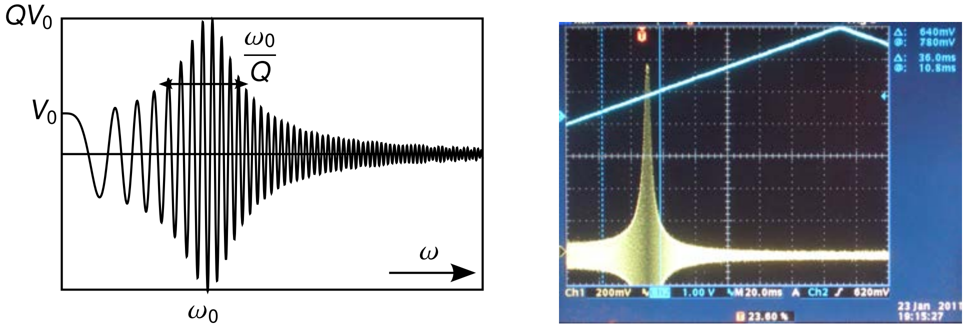


Figure 2: Driving the LCR circuit with a chirped (increasing frequency) driving voltage and observing the voltage across the capacitor with an oscilloscope (left: sketch, right: actual).

Transients

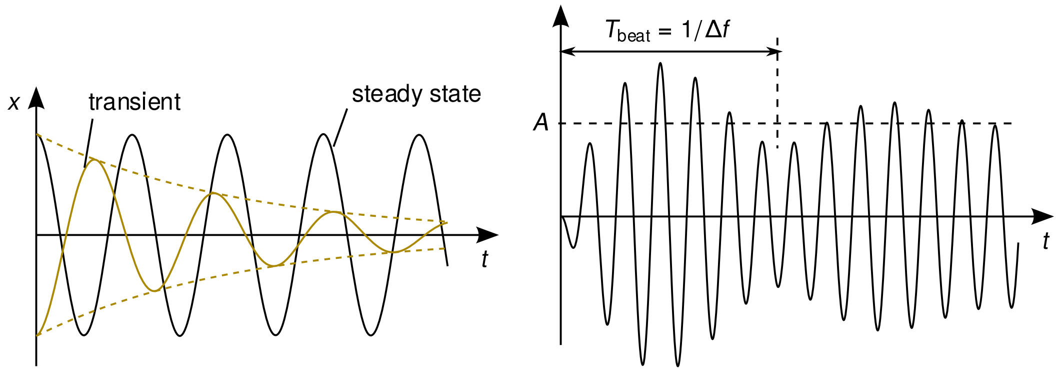
If the equation of motion for the forced oscillator is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

When the force term is removed this becomes the homogeneous equation (i.e., RHS=0), which is just the equation of motion for the free damped oscillator. We can add any solution of the homogeneous equation to the steady state forced oscillator solution and the result will still satisfy the forced oscillator equation. The general solution is then

$$x = \underbrace{A\cos(\omega t + \phi)}_{\text{steady state}} + \underbrace{A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t - \alpha)}_{\text{transient}}$$

(where we have renamed the adjustable constants from the damped oscillator solution from lecture 3 to avoid conflict.) The adjustable parameters A_0 and α are determined by initial conditions ($A_0 = A$, $\alpha = -\phi$ to make the initial displacement zero.)



(sketches not to the same scale)

This is a beating phenomena with beat frequency $\Delta f = |f_d - f| = |\omega_d - \omega|/(2\pi)$.

Power Dissipation and Absorption

The work done by the oscillation against the damping force is

$$dW = -\vec{F}_d.d\vec{x} \qquad \text{Power, } P = \frac{dW}{dt} = -\vec{F}_d.\vec{v}$$

Substituting in for the velocity, $\dot{x} = -\omega A\sin(\omega t + \phi)$ and using $F_d = -b\vec{v}$ gives

$$P = b\vec{v}.\vec{v} = b|v|^2 = b\omega^2 A^2 \sin^2(\omega t + \phi)$$

Taking the time average (over one cycle is sufficient) gives

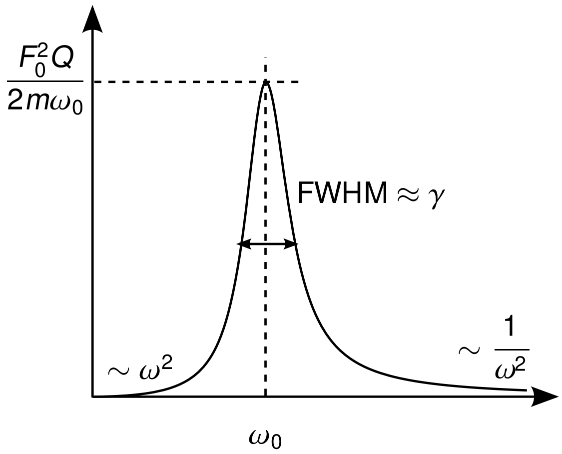
$$P_{av} = \frac{1}{T}\int_t^{t+T} P\,dt = \frac{1}{2}b\omega^2 A^2 = \frac{1}{2}\frac{F_0^2\gamma}{m}\frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

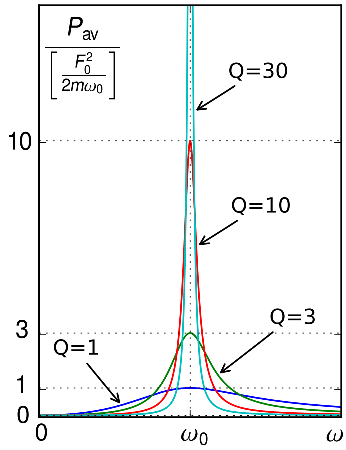
This is equivalent to the work done by the driving force on the system if the system is in a steady state.

$P_a v$ falls to half its maximum value for $(\omega^2 - \omega_0^2)^= \pm\omega\gamma$. For reasonable Q ($\gamma \ll \omega_0$),

$$(\omega^2 - \omega_0^2) = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega\Delta\omega \qquad \Rightarrow \qquad \Delta\omega \approx \pm\frac{\gamma}{2} \qquad \gamma \ll \omega_0$$

The full-width-at-half-maximum FWHM $\approx \gamma$. We therefore call γ the **width**. In terms of the Q-factor, FWHM $\approx \omega_0/Q$.





The function $P_a v$ can be interpreted as a power absorption curve for the system when driven by a force of angular frequency ω. The strength of the absorption peak ($P_a v$ at $\omega = \omega_0$ is proportional to Q). We demonstrated resonance absorption by singing at high-Q wineglasses, observed resonance transfer of oscillations between high-Q wineglasses that had almost the same resonant frequency and observed this did not work when the resonant frequencies were different by much more than γ. We saw the Fraunhofer absorption lines in the solar spectrum as an example of resonance absorption by atoms and ions.

Vibrations and Waves
Formulae Summary : Lectures 8–9

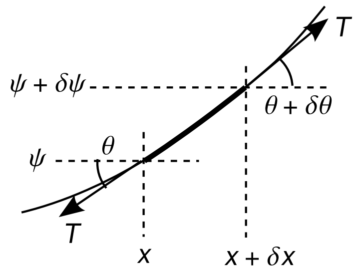
Types of waves

Longitudinal waves The oscillation of the medium is in the same direction as the propagation of the wave. Examples: the N-coupled masses and springs from Lecture 7, sound (pressure waves) in air, seismic P-waves. The wave motion results in compression and rarefaction of the medium. Longitudinal waves can propagate through gasses and liquids.

Transverse waves The oscillation of the medium is transverse to the propagation direction of the wave. Examples: seismic S-waves, waves on a string, the Mexican wave, electromagnetic waves. Transverse waves can exhibit polarization (oscillation can be in more than one direction for a given wave propagation direction). Birefringence is where different polarizations have different propagation speeds.

Some waves can be a combination of longitudinal and transverse: e.g., surface gravity water waves where the motion of the medium (the water) is elliptical having transverse and longitudinal components $90°$ out of phase (i.e., in quadrature).

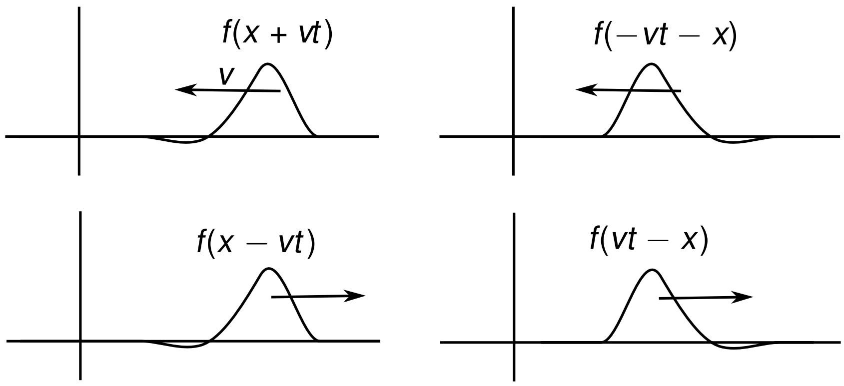
Transverse waves on string



Applying $F = ma$ to a small element of the string for small angles gives the one-dimensional wave equation,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 \psi}{\partial t^2} \qquad \text{— the wave equation}$$

Travelling wave solutions $f(x \pm vt)$ satisfy the wave equation provided $v = \sqrt{T/\mu}$. v is the wave propagation speed. Any function $f(y)$ can be made a solution to the wave equation by the substitution $y = x \pm vt$.



The wave equation is linear: if $f(x \pm vt)$ and $g(x \pm vt)$ are solutions, then so is any superposition of them. The general solution is a superposition of travelling waves in two directions

$$\psi(x,t) = f(x - vt) + g(x + vt)$$

For a travelling wave, the kinetic energy and potential energy are equal to each other at all points along the wave.

Open and Closed Boundaries

At a closed boundary, a travelling wave is reflected and inverted. Example: a travelling wave $f(x - vt)$ travelling towards the end of the string ($x = 0$) which is held fixed. The boundary condition is $\psi = 0$ at the closed end. Writing the reflected wave $g(x + vt)$, then,

$$\psi(0,t) = f(-vt) + g(vt) \qquad \Rightarrow \qquad g(vt) = \cancelto{0}{\psi(0,t)} - f(-vt) \qquad \Rightarrow \qquad g(x+vt) = -f(-x-vt)$$

and we see that the wave is inverted on reflection.

At an open boundary a travelling wave is reflected but not inverted. Example: the end of the string is attached to an idealised massless ring on a frictionless pole. The boundary condition is $\partial\psi/\partial x = 0$ at an open end. Then

$$\frac{\partial\psi}{\partial x}(0,t) = f'(-vt) + g'(vt) = 0 \qquad \Rightarrow \qquad g'(vt) = -f'(-vt)$$

Integrating w.r.t. t gives $g(vt) = f(-vt) \quad \Rightarrow \quad g(x+vt) = f(-x-vt)$, and so the wave is not *not inverted* on reflection.

Harmonic Travelling Waves

A harmonic travelling wave propagating along $+x$:

$$\psi = A\cos\left[\frac{2\pi}{\lambda}(vt - x)\right] \quad = \quad A\cos(\omega t - kx)$$

At any point x, the wave motion is harmonic in time with angular frequency ω.

- λ is the wavelength. $\lambda = v \times$ period $= v(2\pi)/\omega$
- $k = (2\pi)/\lambda$ is the wavenumber.
- $v = \omega/k$ is the speed of propagation (phase velocity).

$\sin(kx \pm kx)$ and $\exp[i(\omega t \pm kx)]$ are equally valid harmonic travelling waves.

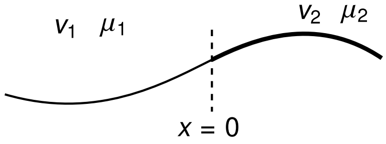
Any wave can be built up as a superposition of travelling waves with different A, ω and k (*Fourier synthesis*).

By considering the beat period between two frequencies f_1 and f_2, $T_{\text{beat}} = 1/\Delta f$, where $\Delta f = |f_2 - f_1|$ we saw qualitatively that the minimum duration of a group or packet of waves was determined by the range of frequencies of the harmonic waves used to make up the wave packet,

$$\Delta t \Delta f \gtrsim 1$$

where Δt is the duration of the wave packet and Δf is the range of frequencies in the wave packet. Using $k = \omega/k$ and $x = vt$ this is readily converted into an equivalent form for a travelling wave $\Delta x \Delta k \gtrsim 1$. This is an example of the Bandwidth Theorem. We also related these inequalities to uncertainty principles of quantum mechanics.

Reflection and transmission at an interface



incident wave ($\rightarrow$) : $A_\mathrm{i}\mathrm{e}^{i(\omega t - k_1 x)}$ transmitted wave ($\rightarrow$) : $A_\mathrm{t}\mathrm{e}^{i(\omega t - k_2 x)}$

reflected wave ($\leftarrow$) : $A_\mathrm{r}\mathrm{e}^{i(\omega t + k_1 x)}$

Boundary conditions:

1. ψ is continuous (string is not broken)
2. $\partial\psi/\partial x$ is continuous (interface does not experience infinite acceleration).

Applying the boundary conditions at the interface gives

BC 1: $$A_\mathrm{i} + A_\mathrm{r} = A_\mathrm{t}$$

BC 2: $$-k_1 A_\mathrm{i} + k_1 A_\mathrm{r} = -k_2 A_\mathrm{t}$$

giving,

$$A_\mathrm{r} = \frac{k_1 - k_2}{k_1 + k_2} A_\mathrm{i} \qquad A_\mathrm{t} = \frac{2k_1}{k_1 + k_2} A_\mathrm{i}$$

Using $v = \omega/k$,

$$A_\mathrm{r} = \frac{v_2 - v_1}{v_2 + v_1} A_\mathrm{i} \qquad A_\mathrm{t} = \frac{2v_2}{v_2 + v_1} A_\mathrm{i}$$

Compare with light at normal incidence on a boundary between two dielectric media, in which case,

$$A_\mathrm{r} = \frac{n_1 - n_2}{n_1 + n_2} A_\mathrm{i} \qquad A_\mathrm{t} = \frac{2n_1}{n_1 + n_2} A_\mathrm{i}$$

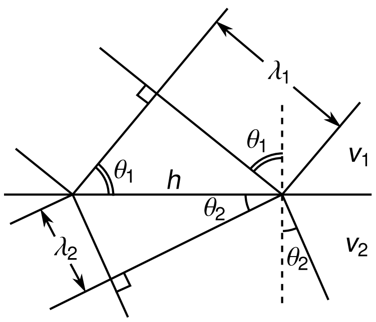
where n_1 and n_2 are the refractive indices of the media ($n_1 = c/v_1$ etc.)

2-D and 3-D waves

In 2-D and 3-D, wavefronts are respectively lines and surfaces of constant phase. Far from a point source, these wavefronts are approximately flat. We call these waves plane waves. The wavefronts are normal to the direction of propagation, have spacing λ and move along with the wave at the velocity of propagation. At a distance r from a point source,

- 2-D waves: Energy density $\propto 1/r$ (e.g., ripples on a pond)
- 3-D waves : Energy density $\propto 1/r^2$

Refraction



Snell's Law:
From continuity considerations (i.e., wavefronts must align on both sides of the interface),

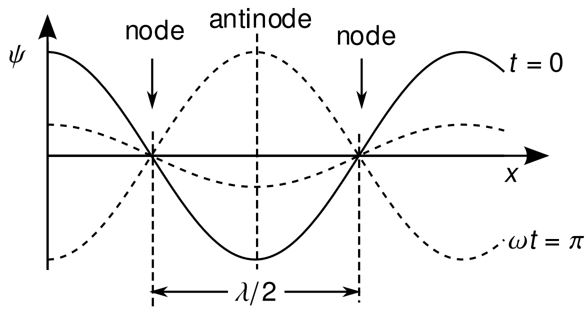
$$\frac{\lambda_2}{\lambda_1} = \frac{h\sin\theta_2}{h\sin\theta_1} \qquad \Rightarrow \qquad \frac{v_2}{v_1} = \frac{\sin\theta_2}{\sin\theta_1}$$

Vibrations and Waves
Formulae Summary : Lectures 10

Standing Waves

The superposition of two equal but counter-propagating travelling waves:

$$\begin{aligned}\psi &= \cos(\omega t - kx) + \cos(\omega t + kx) \\ &= 2\underbrace{\cos\omega t}_{\text{time}}\underbrace{\cos kx}_{\text{space}} \quad \text{— standing wave, (also a normal mode)}\end{aligned}$$



Note: $\cos(\omega t + \phi)\sin kx$ and $\cos(\omega t + \phi)\cos kx$ (or a superposition with the same ω and ϕ) are equally valid standing wave solutions.

Energy

Consider the standing wave $\psi = A\cos kx\cos\omega t$ on a string, At $\omega t = \pi/2$, $\psi = 0$ and so all the standing wave energy is in kinetic form in the transverse motion of the string. The transverse velocity of the string (not to be confused with the wave velocity) is $v_t = \partial\psi/\partial t = -A\omega\cos kx\sin\omega t$, and so the energy per wavelength of the standing wave is

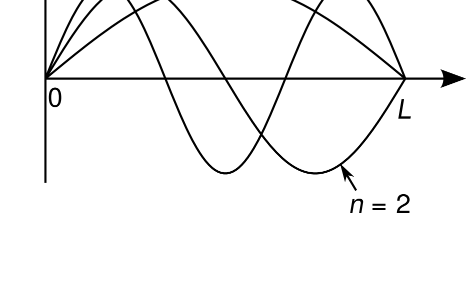
$$E = \text{KE} = \frac{1}{2}mv_t^2 = \frac{1}{2}\int_0^\lambda \mu \mathrm{d}x A^2\omega^2\cos^2 kx = \frac{\mu A^2\omega^2}{2}\overbrace{\int_0^\lambda \cos^2 kx \mathrm{d}x}^{\lambda/2} = \frac{\mu A^2\omega^2\lambda}{4}$$

Using $v = \omega/k = \sqrt{T/\mu}$ and $k = 2\pi/\lambda$, this can can be written as

$$E = \frac{T\pi^2 A^2}{\lambda} \quad \text{(per wavelength)}$$

Standing wave modes

Closed–Closed boundary conditions: For a finite length with closed boundary conditions at $x = 0$ and $x = L$, only discrete values of k are allowed.



The resulting standing wave modes are

$$\psi_n = A_n \sin k_n x \cos \omega_n t \qquad \text{where} \quad k_n = \frac{n\pi}{L}, \quad \omega_n = \frac{n\pi v}{L} = n\omega_1, \quad n = 1, 2, 3\ldots$$

$n = 1$ corresponds to the *first harmonic* or the *fundamental*, $n = 2$ to the second harmonic and so forth. The general situation is a superposition of these normal modes.

Closed–Open boundary conditions: For a closed–open system, the allowed modes have

$$k_n = \frac{(2n-1)\pi}{2L}, \quad \omega_n = \frac{(2n-1)v\pi}{2L} \quad \left(\lambda_n = \frac{4L}{2n-1}, \quad f_n = \frac{(2n-1)v}{4L}\right) \quad n = 1, 2, 3, \ldots$$

In this case the frequencies of the modes have ratios $1 : 3 : 5 : \ldots$.

Musical Instruments: Stringed instruments such as piano, violin, guitar, have closed–closed boundaries. Many wind instruments have closed–open boundaries (e.g., organ, clarinet, trumpet.) The characteristic sounds of instruments are determined by the mix of harmonics that make up the sound.

2D wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \quad \text{(1-D)} \qquad \rightarrow \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \quad \text{(2-D)} \quad \textit{(stated without proof)}$$

The 2D wave equation has harmonic solutions

$$\psi = A\mathrm{e}^{\mathrm{i}(\omega t - k_x x - k_y y)} \qquad \text{where} \quad k_x^2 + k_y^2 = \frac{1}{v^2}\omega^2$$

A system described by the 2-D wave equation, will also exhibit travelling waves and standing waves and a bounded system will have normal modes. The system of modes will in general be more complicated than for the 1-D string. Examples include the sound boards of musical instruments such as the guitar and violin, which are responsible for radiating the sound of the instrument. The nodal-points of the 1-D case are replaced by nodal-lines in the 2-D case.

Vibrations and Waves
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Dispersion

Non-dispersive waves have the same propagation speed v for all k and ω. Examples include electromagnetic waves in vacuum (but not in media), sound, and the transverse waves on the ideal string. When v varies with k, the system is dispersive. Example: the guitar string is non-ideal and has a finite stiffness, which gives an extra term to the wave equation

$$\underbrace{\mu\frac{\partial^2\psi}{\partial t^2} = T\frac{\partial^2\psi}{\partial x^2}}_{\text{non-dispersive}} - \underbrace{C\frac{\partial^4\psi}{\partial x^4}}_{\text{stiffness term}} \qquad \textit{(stated without proof)}$$

Substituting $\psi = \mathrm{e}^{\mathrm{i}(\omega t - kx)}$ (the same harmonic wave solution as before) gives

$$\omega^2 = \underbrace{\frac{T}{\mu}k^2}_{\text{non-disp.}}\left[1 + \underbrace{\frac{C}{T}k^2}_{\text{correction}}\right]$$

The relationship between ω and k is the dispersion relationship. (For the guitar string $C/T \ll 1$ and the dispersive correction is very small, corresponding to a fraction of $1\,\mathrm{Hz}$ over the typical range of frequencies used in the guitar.) Other examples of dispersive systems include water waves, and electromagnetic waves in media.

Wave packets, Phase velocity and Group velocity

Consider the superposition of two different harmonic waves

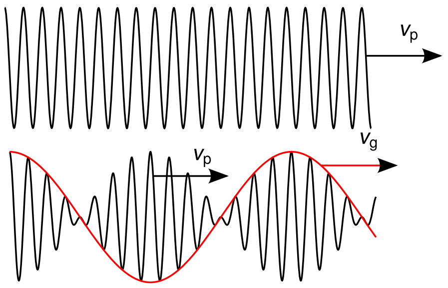
$$\psi_1 = \mathrm{e}^{\mathrm{i}(\omega_1 t - k_1 x)} \quad (v_1 = \omega_1/k_1) \qquad \text{and} \qquad \psi_2 = \mathrm{e}^{\mathrm{i}(\omega_2 t - k_2 x)} \quad (v_2 = \omega_2/k_2)$$

and using the same trignometric identity as when we derived beating,

$$\psi_1 + \psi_2 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right)} \cos\left[\frac{\omega_1 - \omega_2}{2}t - \frac{k_1 - k_2}{2}x\right]$$

$$= 2\underbrace{\mathrm{e}^{\mathrm{i}(\omega t - kx)}}_{v=\omega/k}\underbrace{\cos\left[\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right]}_{v=\Delta\omega/\Delta k} \qquad \left[\text{writing } \frac{\omega_1+\omega_2}{2} = \omega \text{ and } \frac{k_1+k_2}{2} = k\right]$$

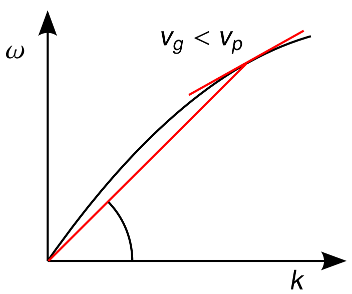
The result is a travelling harmonic wave $\mathrm{e}^{\mathrm{i}(\cdot)}$ with speed ω/k modulated by an envelope $\cos[\cdot]$ which is also a travelling wave with speed $\Delta\omega/\Delta k$.

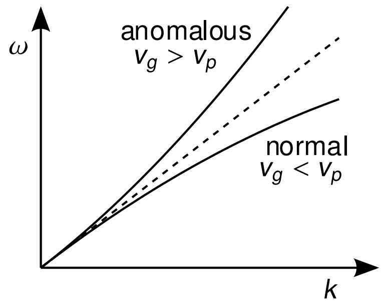


Define:

Phase velocity $v_p = \dfrac{\omega}{k}$

Group velocity $v_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$





- **Normal** dispersion: $v_g < v_p$.
- **Anomalous** dispersion: $v_g > v_p$.

A wave packet (or wave group) is made by Fourier superposition of of harmonic waves from a range of ω and k. The phase velocity gives the speed of the underlying harmonic waves, whereas the group velocity gives the speed of the envelope of the group.

Examples

For transverse waves on a uniform rod, (e.g., the central conducting rail on the London Underground railway track)

$$\mu \frac{\partial^2 \psi}{\partial t^2} = -EI \frac{\partial^4 \psi}{\partial x^4} \qquad \text{(stated without proof)}$$

where E is the Young's modulus and I is the area cross-section moment of inertia. (Note: this case is related to the guitar string except that the stiffness term dominates and the tension term is negligible). Substituting in a harmonic wave trial solution gives the dispersion relation:

$$\psi = \mathrm{e}^{\mathrm{i}(\omega t - kx)} \qquad \Rightarrow \qquad \omega^2 = \frac{EI}{\mu} k^4 \quad \Rightarrow \quad \omega = \sqrt{\frac{EI}{\mu}} k^2$$

This is an example of anomalous dispersion. (It is straightforward to show that $v_g = 2v_p$.)

We also saw examples where the group velocity is much smaller than the phase velocity and where the phase velocity approaches infinity, exceeding c the speed of light in vacuum. We reconciled this by noting that a single harmonic wave cannot carry information—a wave packet or group is required to transmit information (at the group velocity).

Group velocity dispersion

From the bandwidth theorem, a finite length wave packet must consist of harmonic components from a range wavenumbers Δk. If $\mathrm{d}^2\omega/\mathrm{d}k^2 \neq 0$ the group velocity will vary with k. Different parts of the wave group will propagate at different velocities and so the group of waves will not preserve its shape on propagation. A measure of the range of group velocities present in the group is

$$\Delta v_g \approx \frac{\mathrm{d}v_g}{\mathrm{d}k} \Delta k = \frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} \Delta k$$

Consequences of group velocity dispersion are chromatic dispersion and chromatic aberration (the spreading of light with different wavelengths in lenses and prisms), and the lengthening of short laser pulses.

Finally, we saw that in optics there are small regions of anomalous dispersion that occur close to resonant frequencies of some media in which both the phase velocity AND the group velocity may exceed c, the speed of light in vacuum!

Vibrations and Waves
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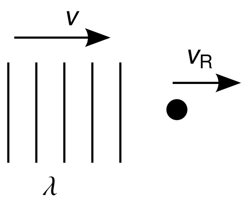
Doppler Effect (Sound)

Start with $v = \lambda f$. This also applies for a moving observer if the wave propagation speed v is replaced with the propagation speed relative to the observer and f with the frequency observed. (The wavelength λ is the same for a stationary or moving observer.)

Moving receiver

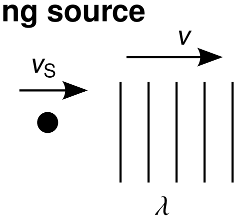


f is the frequency in the medium
f_R is the frequency at the receiver

$$v \to v - v_R, \quad f \to f_R \quad \Rightarrow \quad v - v_R = \lambda f_R$$

$$\text{Using } \lambda = \frac{v}{f} \quad \Rightarrow \quad f_R = \left(\frac{v - v_R}{v}\right) f$$

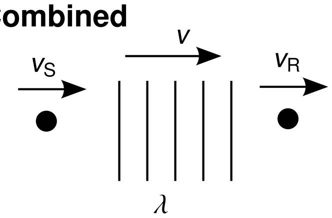
Moving source



$$v \to v - v_S, \quad f \to f_S \quad \Rightarrow \quad v - v_S = \lambda f_S$$

$$\text{Using } \lambda = \frac{v}{f} \quad \Rightarrow \quad \Rightarrow \quad f = \left(\frac{v}{v - v_S}\right) f_S$$

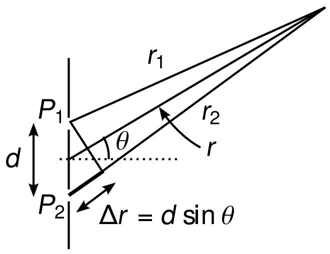
Combined



Combining the two results,

$$f_R = \left(\frac{v - v_R}{v - v_S}\right) f_S$$

Two-slit diffraction



$$r_2 - r_1 = \Delta r = \begin{cases} n\lambda & \text{Constructive interference} \\ (n + \frac{1}{2})\lambda & \text{Destructive interference} \end{cases}$$

For small θ, the angular spacing between diffraction peaks (constructive interference) is

$$\Delta\theta \approx \frac{\lambda}{d}$$

This is the diffraction scale and more generally it gives the typical angular scale for diffraction of light with wavelength λ from a structure of characteristic spatial size d. This is worth remembering.

Intensity

We define time-averaged intensity of a travelling wave (sometimes referred to simply as intensity) as the time-averaged power per unit area over a surface normal to the direction of propagation. Recall that energy density of a wave is proportional to the amplitude squared, so the time-averaged intensity obeys

$$I \propto A^2 \propto |\tilde{\psi}|^2$$

(since for the complex representation $\tilde{\psi} = A\mathrm{e}^{\mathrm{i}(\omega t - kx + \phi)}$, $A = |\tilde{\psi}|$).

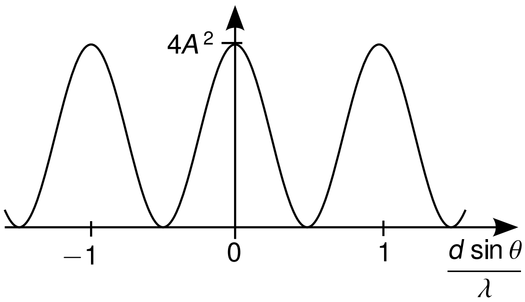
Interference Fringes

From P_1: $\psi_1 = A\mathrm{e}^{\mathrm{i}(\omega t - kr_1)}$ From P_2: $\psi_2 = A\mathrm{e}^{\mathrm{i}(\omega t - kr_2)}$
The linear superposition gives

$$\psi = \psi_1 + \psi_2 = A\mathrm{e}^{\mathrm{i}\omega t}(\mathrm{e}^{-\mathrm{i}kr_1} + \mathrm{e}^{-\mathrm{i}kr_2}) = 2A \underbrace{\mathrm{e}^{\mathrm{i}(\omega t - kr)}}_{\text{travelling wave}} \underbrace{\cos\left(k\frac{\Delta r}{2}\right)}_{\text{envelope}} \qquad \text{where } r = \frac{r_1 + r_2}{2}$$

The intensity is

$$I \propto |\psi|^2 = 4A^2\cos^2\left(k\frac{\Delta r}{2}\right) = 4A^2 \underbrace{\cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)}_{\text{“}\cos^2\text{-fringes”}}$$



Note for two slits, the peak intensity of the fringes is $4\times$ the intensity due to one slit.