Differential Equations

Contents

0.1 Syllabus

Aims To introduce a range of techniques for solving ordinary and partial differential equations and eigenvalue problems. To provide practice in applying these techniques.

Objectives On completion of this course, students will:

- be able to classify differential equations as ordinary or partial, as linear or non-linear, as homogeneous or inhomogeneous, and by their order,
- be able to solve linear 1st order ordinary differential equations (ODEs) by means of an integrating factor,
- be able to recognise and solve separable 1st order ODEs,
- be able to apply the method of variation of parameters to solve differential equations,
- be able to solve linear 2nd order ODEs with constant coefficients by finding the complemen-tary function and a particular integral,
- be able to assess whether two functions are independent by evaluating the Wronskian,
- be able to solve particular examples of 2nd order ODEs with variable coefficients by means of a power series or generalized power series (Frobenius method),
- be able to determine the radius of convergence of a power series,
- have encountered Legendre's equation and Bessel's equation,
- be able to identify separable partial differential equations, including the diffusion, wave, Poisson, and Laplace equations, and to apply the technique of separation of variables,
- understand how boundary conditions may restrict the possible values of the separation constant to a discrete set of eigenvalues,
- understand how to use the eigenfunctions corresponding to these eigenvalues to construct a series solution that satisfies the boundary conditions,
- know what is meant by a Sturm-Liouville equation,
- know that the eigenvalues of a Sturm-Liouville equation are real and the eigenfunctions are orthogonal.

1 Introduction

Differential equations can be classified by four properties: *ordinary* or *partial*, *order*, *linear* or *non-linear*, *homogeneous* or *inhomogeneous*.

1.0.1 Ordinary or Partial?

If the solution of the differential equation has the form,

$$y = f(x)$$

where the solution has one dependent and one independent variable, it is an *ordindary differential equation* or *ODE*. Conversely, if the solution has more than one independent variable it is a *partial differential equation* or *PDE*.

1.0.2 Order

The order of a differential equation is the order of the highest derivative in the equation, i.e. the number of times it has been differentiated. Example:

$$\frac{d^2y}{dx^2} - y = 0$$

is a second order differential equation, as the highest derivative has been differentiated twice (with respect to x).

1.0.3 Linearity

We can write ODEs as an operator, i.e. $\hat{O}y = f(x)$. The ODE is linear if:

- $\hat{O}(y_1 + y_1) = \hat{O}y_1 + \hat{O}y_2$
- All terms of y and its derivatives are of first degree and seperate.

If the ODE is linear, then we will write the operator as $\hat{L}$.

1.0.4 Homogeneity

If we consider a differential equation for example a simple first order ODE,

$$\frac{dy}{dx} = f(x)$$

then the differential equation is homogeneous if $f(x) = 0$, i.e. all functions are in terms of y or its derivatives. In the opposite case, $f(x) \neq 0$, the equation is imhomogeneous.

2 First Order Ordinary Differential Equations

2.1 Separation of Variables

For equation of the form

$$\frac{dy}{dx} = f(y)g(x)$$

we can re-arrange this equation to give:

$$\frac{dy}{f(y)} = g(x)dx$$

and by carrying out the integration the solution to the equation, $y(x)$ can be found. In some cases it is possible to make the equation seperable by a simple change of variables, even if at first the equation is not seperable.

2.2 General Solution

Equations of this form can be solved by multiplying through with an *integrating factor* $\mu(x)$

$$a_1(x)\frac{dy}{dx} + a_0(x)y = h(x)$$

Summary To solve a first order linear ODE with an integrating factor:

1. Re-arrange the ODE into the form,

$$\frac{dy}{dx} + f(x)y = g(x) \tag{2.1}$$

2. Evaluate the integrating factor,

$$\mu(x) = e^{\int f(x)\,dx} \tag{2.2}$$

3. Multiply through by the multiplying factor,

$$\mu(x)\frac{dy}{dx} + \mu(x)f(x)y = \mu(x)g(x)$$

4. Now the solution is

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)dx \tag{2.3}$$

2.3 Existence and Uniqueness - Initial Value Problem

Consider a linear first order ODE of the form,

$$\frac{dy}{dx} + f(x)y = g(x)$$

where $y(x_0) = y_0$.

> **Theorem** If $f(x)$ and $g(x)$ are continuous over an interval $\alpha \leq x \leq \beta$ which contains x_0 then there exists a unique function which satisfies the differential equation and the intial conditions over the interval.

For non-linear equations the conditions are more restrictive.

2.4 Introduction To The Method of Variation of Parameters

The basic idea behind the method of variation of parameters - to solve a difficult differential equation, find the solution $u(x)$ to a simpler but similar problem then try $u(x)v(x)$ as a solution to the original differential equation and solve for v.

Consider an inhomogeneous linear first order ODE,

$$\frac{dy}{dx} + f(x)y = g(x) \tag{2.4}$$

If we solve the homogenous 'version' of this equation ($g(x) = 0$) then we have a solution,

$$y(x) = Ae^{-\int f(x)\,dx}$$

Note - this is $1/\mu$. If we let

$$\frac{du(x)}{dx} = -f(x)u(x) \tag{2.5}$$

$$y(x) = u(x)v(x) \tag{2.6}$$

then substitute (2.6) into the original, inhomogeneous ODE (2.4),

$$\frac{du}{dx}v + u\frac{dv}{dx} + fuv = g(x)$$

$$\therefore \frac{dv}{dx} = \frac{g(x)}{u(x)} \Rightarrow v(x) = \int \frac{g(x)}{u(x)}dx$$

Notice that

$$y(x) = u(x)\int \frac{g(x)}{u(x)} = \frac{1}{\mu(x)}\int \mu(x)g(x)dx$$

3 Second Order and Higher Linear ODEs With Constant Coefficients

3.1 General Properties of Solutions of Second Order Linear ODEs

3.1.1 Existence and Uniqueness

Consider the general form of a second order linear ODE,

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{3.1}$$

Rewritten,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = h(x) \tag{3.2}$$

This is a linear differential equation so we can write $\hat{L}y = h(x)$. We have the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$, where y' represents the first derivative.

Theorem If $p(x)$, $q(x)$ and $h(x)$ are continuous over an interval $\alpha \leq x \leq \beta$ which contains x_0 then there exists a unique solution.

A general method of solution to second order ODEs only exists for the case of a_0, a_1 and a_2 are *constant* (not functions of x).

3.1.2 Form of The General Solution

The general solution of a second order ODE with constant coefficients is equal to the *sum of the complementary function and the particular integral.*

Proof Consider two particular integrals $P_1$and P_2 which are solutions of a second order linear ODE of form (3.1),

$$\hat{L}P_1(x) = f(x) \qquad\qquad \hat{L}P_2(x) = f(x)$$

where $\hat{L}$ is an operator representing the differential equation. We can show

$$\hat{L}P_1 - \hat{L}P_2 = \hat{L}\left(P_1 + P_2\right) = 0$$

i.e. $P_1 + P_2$ is a solution of the homegeneous equation. The solution of the homogeneous equation is the complemtary function, hence

$$P_1 + P_2 = y_c$$

where y_c is the complementary function. It is then trivial to show that

$$P_2 = y_c + P_1$$

hence P_2 is a general solution of the differential equation.

Summary To find the solution to a second order ODE with constant coefficients:

1. Find the complementary function.
2. Find any particular integral.
3. The general solution is the sum of the complementary function and particular integral.
4. Use initial conditions to solve for arbitrary constants.

3.2 Complementary Function

Consider the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

We can factorise this equations as

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 2\right) y = 0$$

This is true for any $(d/dx - a)\,(d/dx - b)$, as long as the coefficients of the differential equation are constant. The equation is zero if

$$\frac{dy}{dx} - y = 0 \quad \text{or} \quad \frac{dy}{dx} + 2y = 0$$

These are seperable equations with solutions

$$y = C_1 e^x; \quad y = C_2 e^{-2x}$$

Now if $(d/dx + 2)y = 0$, then

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 2\right) y = \left(\frac{d}{dx} - 1\right) \cdot 0 = 0$$

So any solution of $(d/dx + 2)y = 0$ is a solution of the full differential equation, and similarly any solution of $(d/dx - 1)y = 0$ is also a solution. Since the two solutions are linearly independent, a linear combination of them is the general solution.

So for the general case of finding the complementary function for a liner second order ODE with constant coefficients

Summary To find the complementary function:

1. Write the differential equation as

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

2. Solve the characteristic equation

$$a_2 m^2 + a_1 m + a_0 = 0$$

which is just a standard quadratic polynomial and hence can be solved with the quadratic formula:

$$m_{\pm} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

3. The complementary function for two real roots is

$$y_c(x) = C_1 e^{m_+ x} + C_2 e^{m_- x}$$

for two complex roots

$$\begin{aligned} & y_c(x) = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ \text{or} \quad & y_c(x) = e^{\alpha x}(C_3 \cos \beta x + C_4 \sin \beta x) \end{aligned}$$

and for repeated roots

$$y_c(x) = (C_1 + C_2 x)\, e^{mx}$$

The discriminant of the quadratic formula also reveals more behaviour of the solution:

- $a_1^2 - 4a_2 a_0 > 1$ - two real roots, this is known as *overdamped*.

- $a_1^2 - 4a_2a_0 < 1$ - two complex roots, this is known as *underdamped.*
- $a_1^2 - 4a_2a_0 = 0$ - repeated roots, known as *critically damped.*

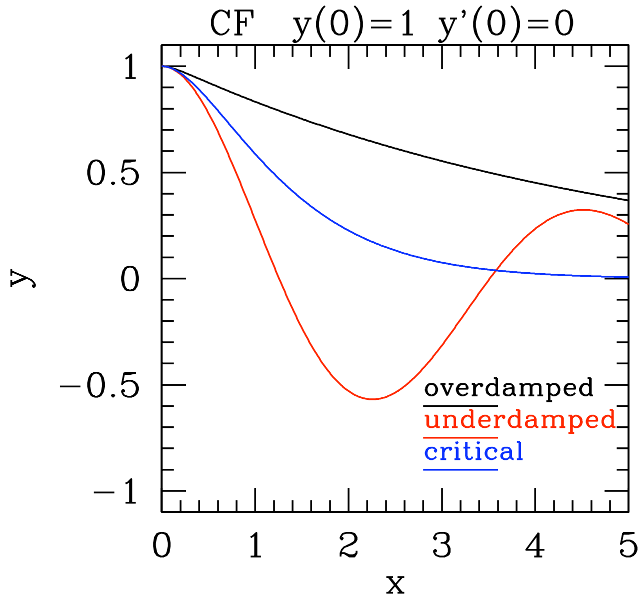

Figure 3.1: A plot of the complementary functions for the cases of overdamped, underdamped and critically damped.

3.3 The Particular Integral

We need to find a particular solution of the inhomogeneous equation $\hat{L}y = f(x)$ which has no arbitrary constants, there are several methods of doing so.

3.3.1 Inspection

If there is a very simple particular solution we may be able to guess and verify it. Consider the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 5$$

It is easy to see that for constant y, y'' and y' are zero, so $y_p = 5/3$.

3.3.2 Successive Integration of Two First-Order Equations

This is a straight-forward method which can always be used to solve equations of the form (3.1). In practice however, it often involves more work than various special methods.

Consider the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x \tag{3.3}$$

which we can factorise and write as

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 2\right) y = e^x$$

Now if we let

$$u = \left(\frac{d}{dx} + 2\right) y$$

then the differential equation (3.7) becomes

$$\left(\frac{d}{dx} - 1\right) u = e^x \quad or \quad \frac{du}{dx} - u = e^x \tag{3.4}$$

This is a linear first order ODE which we can solve as we did in Section 2: finding the integrating factor and retrieving the solution

$$\mu(x) = e^{\int -dx} = e^{-x}$$

So the solution of (3.8) is

$$u(x) = \frac{1}{\mu}\int e^{-x}e^x dx = xe^x + c_1e^x$$

From this solution (3.7) becomes

$$\left(\frac{d}{dx} + 2\right) y = xe^x + c_1e^x \quad or \quad \frac{dy}{dx} + 2y = xe^x + c_1e^x$$

This is another linear first order ODE which was can solve as before

$$\mu(x) = e^{2x}$$

$$\begin{aligned} ye^{2x} &= \int xe^{3x} + c_1e^{3x}dx \\ &= \frac{1}{3}xe^3 - \frac{1}{9}e^{3x} + \frac{1}{3}c_1e^{3x} + c_2 \end{aligned}$$

$$y(x) = \frac{1}{3}xe^x + c_1'e^x + c_2e^{-2x}$$

Notice that we have obtained the general solution all in one process. We could have just found the particular integral $xe^x/3$ by omitting the arbitray constants at each step of the intergration, as these formed the complementary function and also dropping terms already in the complementary function. Since it is easier to write the complementary function, it saves time to omit those terms when finding the particular integral.

3.3.3 Exponential Right-Hand Side

Let us consider how to find a particular integral when the right hand side of (3.1) is $f(x) = ke^{cx}$, where k and c are given constants. Consider that c may be complex. Let a and b be the roots of the factorised equation such that

$$\left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right) y = ke^{cx} \tag{3.5}$$

Solving (3.9) by successive integration is straightforwad and gives the result that the particular integral is a multiple of e^{cx}. Now that we know the form of the particular integral, we simply assume a solution of this form and solve for the constant.

For example, solving the equation

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 5\right) y = 7e^{2x} \tag{3.6}$$

If we substitute our general form the particular integral, $y_p = Ce^{2x}$

$$\frac{d^2y_p}{dx^2} + 4\frac{dy_p}{dx} - 5y_p = 7e^{2x}$$

$$C(4e^{2x} + 8e^{2x} - 5e^{2x}) = 7e^{2x}$$

And hence C must have a value of 1 in this case, and we have a general solution (including complementary function)

$$y(x) = C_1e^x + C_2e^{-5x} + e^{2x}$$

In solving (3.7) we have seen that if c is equal to either a or b then we have a particular integral of the form Cxe^{cx} and in (3.10) that for $c \neq a$ or b then $y_p = Ce^{cx}$. These rules can be summed up as:

Summary To find a particular integral of a DE with exponential right hand side:

1. Consider the form of the differential equation

$$\left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right) y = ke^{cx}$$

   by simple factorisation or solving the characteristic equation.

2. The particular integral's form is then

$$y_p = \begin{cases} Ce^{cx} & \text{if } c \text{ is not equal to either } a \text{ or } b; \\ Cxe^{cx} & \text{if } c \text{ is equal to } a \text{ or } b,\ a \neq b; \\ Cx^2e^{cx} & \text{if } c = a = b. \end{cases}$$

   where C is an undetermined coefficient, to be determined by sustitution into the differential equation.

3.3.4 Use of Complex Exponentials

In applied problems, $f(x)$ is very often a sine or cosine representing alternating emf, periodic force etc. We could find y_p by the method of successive integration or we could replace the sine or cosine with a complex exponential.

For example, we have the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4\sin 2x$$

Instead of going in feet first with the successive integration method, we will first solve the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4e^{2ix}$$

Taking a particular integral of $Y_p = Ce^{2ix}$ and subbing into the differential equation

$$\frac{d^2Y_p}{dx^2} + \frac{dY_p}{dx} - 2Y_p = Ce^{2ix}(-4 + 2i - 2) = 4e^{2ix}$$

Hence

$$C = \frac{4}{2i - 6} = -\frac{1}{5}(i + 3)$$

Now taking the imaginary (as our initial DE had a sine function) part of $Y_p = -\frac{1}{5}(i+3)e^{2ix}$ we get

$$y_p = -\frac{1}{5}\cos(2x) - \frac{3}{5}\sin 2x$$

So we can summarise the use of complex exponentials as:

Summary To find a particular integral of a DE with a sine or cosine right hand side:

1. Solve
$$\left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right) y = ke^{i\alpha x}$$
2. Take the real or imaginary part as required.

3.3.5 Method of Undetermined Coefficients

The method of assuming a certain form of solution and then determing the constant factor C is an example of the method of undetermined coefficients. It is straight forward to find the corresponding result to the use of complex exponentials for the case of $f(x)$ is a polynomial or a polynomial times an exponential.

Summary To find a particular integral of a DE with polynomial (in x) RHS of order n.

1. Try a solution of the form
$$y_p = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_0 x^0$$
2. Find the coefficients A_n.

Summary To find a particular integral of a polynomial times an exponential $e^{cx}P_n(x)$, where $P_n(x)$ is a polynomial of degree n:

1. Consider the factorised form of the differential equation

$$\left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right) y = e^{cx} P_n(x)$$

2. The particular integral is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if } c \text{ is not equal to either } a \text{ or } b; \\ xe^{cx}Q_n(x) & \text{if } c \text{ is equal to } a \text{ or } b,\ a \neq b; \\ x^2e^{cx}Q_n(x) & \text{if } c = a = b. \end{cases}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with undetermined coefficients to be found to satisfy the differential equation. Note that sines and cosines are included in this case by the use of complex exponentials as before. This is just a generalised version of the previous method.

Warning! If the right hand side ($f(x)$) of the equation is in the CF try $y_p = Cx^n f(x)$ or Lagrange's method of variation of parameters.

3.4 Independence of Functions and the Wronskian

By a definition similar to that of vectors, we say that the functions $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ are linearly dependent if some linear combination of them is identically zero, that is, if there are constants $k_1,\ k_2, \ldots,\ k_n$, not all zero, such that

$$k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x) = 0$$

For example, $\sin^2 x$ and $\cos^2 x$ are linearly dependent since

$$\sin^2 x - (1 - \cos^2 x) \equiv 0$$

but $\sin x$ and $\cos x$ are not, as there is no linear combination which will give zero for all x.

If we want to know if given set of function is linearly independent then we can use the Wronskian:

If $f_1(x)$, $f_2(x), \ldots, f_n(x)$ have derivatives of order $n-1$ and if the determinant

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix} \neq 0$$

then the functions are linearly independent. The determinant, W, is called the Wronskian of the functions.

Any linear combination of solutions is suitable as a basis of a solution of a DE provided they are independent, i.e. the Wronskian is non-zero.

3.5 Higher Order Equations with Constant Coefficients

The same methods for solving the differential equations work, with the general solution $GS = CF + PI$. To obtain the CF we have to solve higher order characteristic equations, i.e.

$$a_3m^3 + a_2m^2 + a_1m + a_0 = 0$$

so we will get three roots of m for a third order DE, so the CF is

$$y_c = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x}$$

To find a particular integral, any of the methods of undetermined coefficients is valid.

4 Series Solutions of Ordinary Differential Equations

We will consider solutions to the general linear second order homogeneous ODE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{4.1}$$

Important examples of equations of this form are:

- *Legendre's Equation* (spherical problems):

  $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$$

  where l is a constant.

- Another example is *Bessel's Equation* (cylindrical problems):

  $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-p^2)y = 0$$

  where p is a constant.

4.1 Radius of Convergence

Consider the partial sum of a series

$$S_N = \sum_{n=1}^{N} t_n$$

where t_n are the terms of the series. If $\lim_{N\to\infty} S_N$ exists, then the series is said to converge. To test if a series converges absolutely, use the ratio test:

$$r_n = \left|\frac{t_{n+1}}{t_n}\right|$$

If $r = \lim_{n\to\infty}$ then if

$$r \begin{cases} < 1 & \text{the series converges absolutely} \\ = 1 & \text{unclear} \\ > 1 & \text{the series diverges} \end{cases}$$

The radius of convergence R is the limiting value of $|x - x_0|$ within which the series converges.

4.2 Power Series Solution

We rewrite equation (4.1) as

$$\frac{d^2y}{dx^2} + \frac{a_1(x)}{a_2(x)}\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

Under what conditions does a power series exist?

$$y(x) = \sum_n C_n(x-x_0)^n$$

We can answer this question by assuming a power series is a solution of the general DE and working backwards. So the initial conditions of the differential equations are

$$\begin{aligned} & y(x_0) = y_0; \qquad y'(x_0) = y_0' \\ & \text{and} \\ & C_0 = y_0; \qquad C_1 = y_0' \end{aligned}$$

Hence

$$\begin{aligned} \frac{d^2y}{dx^2} &= -p(x)\frac{dy}{dx} - q(x)y \\ &= -p(x)[C_1 + 2C_2(x-x_0) + \ldots] - q(x)[C_0 + C_1(x-x_0) + \ldots] \end{aligned}$$

evaluating this expression at $x = x_0$

$$2C_2 = -p(x_0)y_0' - q(x_0)y_0$$

so we require that $y(x_0)$, $y'(x_0)$ exist.

Coupled with the requirement that the radius of convergence of the power series R must be greater than 0 we have the conditions we were looking for: a function is analytic about x_0 if when expressed as a power series

$$f(x) = \sum_n C_n(x-x_0)^n$$

it has a radius of convergence $R > 0$, which requires that $f(x_0)$ and $f'(x_0)$ exist.

> A power series solution exist if $p(x)$ and $q(x)$ are analytic - this is described by saying x_0 is an ordinary point.

Summary To find the power series solution follow the following:

1. Write out each term of the ODE, lining up powers of x.
2. Find the lowest power series of x where all the terms are present and express those terms as series and leave the remaining "orphan" terms,
3. Set coefficients of each power of x to zero and series terms will give the recurrence relation.

Example We illustrate the method of series solution by solving the simple DE

$$\frac{dy}{dx} - 2xy = 0$$

We assume a solution of this differential equation in the form of a power series

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + \dots \\ &= \sum_{n=1}^{\infty} c_n x^n \end{aligned}$$

where the c's are to be found. Differentiating this assumed solution, we get

$$\begin{aligned} \frac{dy}{dx} &= c_1 + 2c_2 x + 3c_3 x^2 + \dots \\ &= \sum_{n=1}^{\infty} n c_n x^{n-1} \end{aligned}$$

We then substitue these two back into the differential equation, which must be satisfied for all x so

$$\begin{aligned} c_1 + 2c_2 x + 3c_3 x^2 + \dots &= 2x(c_0 + c_1 x + c_2 x^2 + \dots) \\ &= 2c_0 x + 2c_1 x^2 + 2c_2 x^3 + \dots \end{aligned}$$

and comparing we see that:

$$c_1 = 0, \quad c_2 = c_0, \quad c_3 = \frac{2}{3} c_1 = 0; \quad c_4 = \frac{1}{2} c_2 = \frac{1}{2} c_0$$

or in general:

$$n c_n = 2 c_{n-2} \;\Rightarrow\; c_n = \begin{cases} 0 & \text{for odd } n \\ \frac{2}{n} c_{n-2} & \text{for even } n \end{cases}$$

Putting $2m = n$ as this solution only allows for even terms we can see

$$c_{2m} = \frac{2}{2m}c_{2m-2} = \frac{1}{m}c_{2m-2} = \frac{1}{m}\frac{1}{m-1}c_{2m-4} = \ldots = \frac{1}{m!}c_0$$

So out power series solution becomes

$$y = c_0 + c_0x^2 + \frac{1}{2!}c_0x^4 + \ldots = c_0\sum_{m=0}^{\infty}\frac{x^{2m}}{m!} = c_0e^{x^2}$$

The latter being the solution we would have obtained from solving this differential equation normally. The radius of convergence in this case in infinite, as the power series of e^x is valid $\forall x$.

We cannot always expect to find the closed form of a power series solution, but in simple cases we may recognise it. Of course in that case, the problem could have been done without power series at all - the real need for power series solutions is for problems with no closed form. Also note that not all solutions have series expansions in powers of x, for example $\ln x$ or $1/x^2$. All we can say is that if there is a solution which can be represented by a convergent power series, this method will find it.

4.2.1 Legendre's Equation and the Legendre Polynomials

Legendre's equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dx}{dy} + l(l+1)y = 0 \tag{4.2}$$

In physical problems we often see this equation with $x = \cos\theta$, hence we are interested in solutions over the range $-1 \leq x \leq 1$. So we define

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dx}{dy} + \left[l(l+1) - \frac{m}{1-x^2}\right]y = 0 \tag{4.3}$$

and in the case of Legendre's equation, $m = 0$. We can rewrite this as

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{k}{1-x^2}y = 0 \tag{4.4}$$

We can see that $p(x) = 2x/(1-x^2)$ and $q(x) = k/(1-x^2)$ are analytic over the range $-1 < x < 1$, so power series solution does exist within these bounds. If we write the power series

$$\begin{aligned} y &= c_0 + c_1x + c_2x^2 + \ldots \\ \frac{dy}{dx} &= c_1 + 2c_2x + 3c_3x^2 + \ldots \\ \frac{d^2y}{dx^2} &= 2c_2 + 6c_3x + 12c_4x^2 + \ldots \end{aligned}$$

So going back to the form of Legendre's equation in (4.3)

$$ky = kc_0 + kc_1 x + kc_2 x^2 + \dots \tag{4.5}$$

$$-2x\frac{dy}{dx} = -2xc_1 - 4c_2 x^2 - 6c_3 x^3 - \dots \tag{4.6}$$

$$\frac{d^2y}{dx^2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots \tag{4.7}$$

$$-x^2\frac{d^2y}{dx^2} = -2c_2 x^2 - 6c_3 x^3 - 12c_4 x^4 \dots \tag{4.8}$$

The sum of (4.5) to (4.8) must be zero:

$$0 = (kc_0 + 2c_2) + x\,(kc_1 + 2c_1 + 6c_3) + x^2(kc_2 - 4c_2 + 12c_4 - 2c_2) + \dots$$

Hence for the odd powers of x

$$c_2 = -\frac{k}{2}c_0 \qquad\qquad c_4 = \frac{(6-k)}{12}c_2$$

and for odd powers of x

$$c_3 = \frac{(2-k)}{6}c_1$$

Examining the pattern closely we see that for x_n we get

$$c_{n+2} = \frac{[n(n+1)-k]}{(n+1)\,(n+2)}c_n$$

and so

$$\begin{aligned} y =& c_0\left[1 - \frac{k}{2!}x^2 - \frac{k\,(6-k)}{4!}x^4 - \frac{k(6-k)(20-k)}{6!}x^6 + \dots\right] \\ &+ c_1\left[x + \frac{2-k}{3!}x^3 + \frac{(2-k)(12-k)}{5!}x^5 + \dots\right] \end{aligned}$$

If we substitute $k = l(l+1)$ back in for Legendre's equation we see the solution is

$$\begin{aligned} y =& c_0\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 + \dots\right] \\ &+ c_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 + \dots\right] \end{aligned}$$

Radius of Convergence We find that for these power series,

$$\lim_{n\to\infty} r_n = x^2$$

so we know that the power series are absolutely convergent for $-1 < x < 1$, but we need a further test to determine the behaviour at $x = \pm 1$. As can be seen from equation (4.4), the $p(x)$ and $q(x)$ have singularities at $x = \pm 1$, so in general the series will not converge at $x = \pm 1$.

Legendre Polynomials In the case of $x = \cos\theta$, we know that the range of x contains ± 1 so we want a solution for all θ. When l is an integer, the solution $P_l(x)$ that is regular at $x = 1$ is also regular at $x = -1$, and the series for this solution terminates (i.e. is a polynomial). For example, $l = 0$: the $c_1$series diverges, but the c_0 series is simply c_0 as every other term contains a factor of l. For $l = 1$, the c_1 series is simple $c_1 x$, as ever other term contains an $l - 1$ factor and the c_0 series diverges. Generally, the Legendre polynomial can be found by:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} [(x^2 - 1)^l]$$

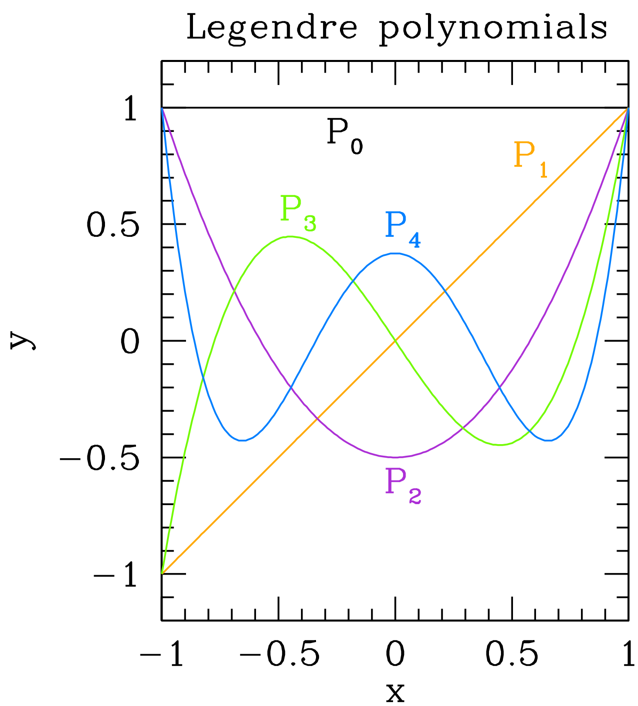

Figure 4.1: A plot of the first 4 Legendre polynomials.

The Legendre polynomials form a complete orthonormal set on the interval $[-1, 1]$, or $[0, \pi]$ for $x = \cos\theta$. The featurs of the Legendre equation are a consequence of the structure of the equation, which is of Sturm-Liouville form (Section 7).

Note:

$$\int_{-1}^{1} P_l(x)^2 dx = \frac{2}{2l + 1}$$

as the P_l are not properly normalised.

The Legendre polynomials are analogous to Fourier series in that any well behaved function on the interval $[-1, 1]$ can be expressed as a linear combination of Legendre polynomials

$$f(x) = \sum_{l=0}^{\infty} C_l P_l(x)$$

where the coeffecients C_l can be found by

$$C_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx$$

Associated Legendre Equations Recall that the equation we solved was a special case of equation (4.3), where $m = 0$. Eq. (4.3) is known as the associated Legendre equation, and its solutions are the associated Legendre polynomials,

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m}[P_l(x)]$$

4.3 Generalised Power Series and the Frobenius Method

The previous method of expanding power series about a point x_0 as a solution to a differential equation is only valid if x_0 is an ordinary point. Some important equations cannot be solved by this method, so a generalised power series is needed.

Frobenius' Theorem If x_0 is a regular singular point, at least one solution in the form of a generalised power series expanded about x_0 exists,

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n x^n$$

The condition $c_0 \neq 0$ leads to the *indicial equation* which is a quadratic order r, usually providing two solutions and two series. Problems arise with the indicial equation when it has equal roots or roots differing by an integer. In these cases we do not need to find a second solution.

4.3.1 Bessel's Equation

Bessel's equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

which we can rewrite as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{(x^2 - p^2)}{x^2}y = 0$$

So $p(x) = 1/x$ and $q(x) = (x^2 - p^2)/x^2$ have singularity at $x = 0$. Solutions can be found in cases where singularities are not "too severe". In this case, the singularity is at $x_0 = 0$ so our series solution is

$$y = \sum_{n=0}^{\infty} c_n x^{r+n} = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \dots$$

And so

$$x^2 y = c_0 x^{r+2} + c_1 x^{r+3} + c_2 x^{r+4} + \dots \tag{4.9}$$
$$-p^2 y = -p^2 c_0 x^r - p^2 c_1 x^{r+1} - p^2 c_2 x^{r+2} + \dots \tag{4.10}$$
$$x\frac{dy}{dx} = r c_0 x^r + (r+1)c_1 x^{r+1} + (r+2)c_2 x^{r+2} + \dots \tag{4.11}$$
$$x^2\frac{d^2y}{dx^2} = r(r-1)c_0 x^r + (r+1)r c_1 x^{r+1} + (r+2)(r+1)c_2 x^{r+2} + \dots \tag{4.12}$$

And we know the sum of equations (4.9) to (4.12) is zero, hence

$$\begin{aligned} 0 &= c_0 x^r (r^2 - p^2) + c_1 x^{r+1}[(r+1)^2 - p^2] \\ &\quad + \sum_{n=2}^{\infty} x^{r+n}\left[c_{n-2} - p^2 c_n + (r+n)c_n + (r+n)(r+n-1)c_n\right] \end{aligned}$$

The c_0 term reveals the individual equation

$$p = \pm r$$

Using the individual equation, we see that c_1 must be zero except when $r = -1/2$ and we find the recurrence relation

$$c_{n-2} = -c_n[(r+n)^2 - p^2]$$

So odd for odd n, c_n is zero and the even n terms form c_0. We have series solution for $+p$ and $-p$, J_p and J_{-p}. The general solution is

$$y = c_1 J_p + c_2 J_{-p}$$

The J_p are Bessel functions of the first kind, order p.

Summary To solve a differential equation with the Frobenius method:

1. Substitute the power series for y about the singular point x_0 into the differential equation.
2. The restriction $C_0 \neq 0$ provides the indicial equation.
3. Find the recursion relations in the terms of the series.
4. Express the general solution as the sum of the power series obtained.

In problem cases - where the indicial equation has equal roots or the roots differ by an integer - take the larger root and find y_1 then try (for example) $y_2 = u(x)y_1$.

5 Partial Differential Equations in Cartesian Co-ordinates

Partial differential equations arise where we have two or more independent variables, e.g. $u(x,t)$ and the simplest possible PDE

$$\frac{\partial u}{\partial x} = 0$$

where the general solution is $u = g(t)$ i.e. arbitrary function.

5.1 Examples From Physics

5.1.1 Laplace and Poisson Equations

The are many physical examples of partial differential equations for example in electromagnetism

$$\nabla^2 \mathbf{V} = -\frac{\rho}{\varepsilon_0}$$

This is an example of Poisson's equation. In the absence of charge the equation becomes

$$\nabla^2 \mathbf{V} = 0$$

or Laplace's equation. Where ∇^2 is the Laplacian operator.

5.1.2 Wave Equations

Consider the wave equation for a taut string

$$\frac{\partial^2 u}{\partial x^2} = \frac{\lambda}{T}\frac{\partial^2 u}{\partial t^2}$$

where λ is the mass per unit length and T is the tension in the string. It can be shown that the dimension of λ/T is 1/velocity^2, i.e. the general wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where v is the speed of propagation. Looking at Maxwell's equations in a vaccuum,

$$\nabla \cdot \mathbf{E} = 0 \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad\qquad \nabla \times \mathbf{B} = \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

Taking the curl of Faraday's Law,

$$\nabla^2\mathbf{E} = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \mu_0\varepsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\text{or } \nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

5.1.3 Diffusion (Heat) Equation

Combines continuity equation with Fick's Law. Let $\mathbf{f}(\mathbf{r}, t)$ describe a flux vector field so that the rate of flow of the quantity (e.g. number N, charge Q, energy U) across an elemental area $d\mathbf{s}$ is $\mathbf{f} \cdot d\mathbf{s}$.

To describe the continuity equation consier an elemental volume with dimension dx, dy, dz with flux into the sides of the volume J_x, J_y, J_z .

The flux out of the volume is then $J_i + \partial_i J_i di$, where $i = x, y, z$. The net flow through the volume is therefore

$$\begin{aligned} dN &= \left(J_x - J_x - \frac{\partial J_x}{\partial x}dx\right) dydzdt \\ &+ \left(J_y - J_y - \frac{\partial J_y}{\partial y}dy\right) dxdzdt \\ &+ \left(J_z - J_z - \frac{\partial J_z}{\partial z}dz\right) dxdydt \\ &= -dxdydzdt\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) \end{aligned}$$

So a $dxdydz = V$, we can write

$$dN = -V dt \nabla \cdot \mathbf{f} \;\; \Rightarrow \;\; \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{f}$$

the continuity equation. In many physical situations we observe Fick's law:

$$\mathbf{f} = -D\nabla\rho$$

and combining this with continuity equation we find

$$\frac{\partial \rho}{\partial t} = -D\nabla^2\rho$$

this diffusion equation.

Apply this now to heat diffusion we use Fourier's law of heat flow

$$\frac{1}{A}\frac{\partial Q}{\partial t} = \mathbf{q} = -k\nabla\underline{\theta}$$

and the continuity equation

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{q}$$

We want to relate u to θ. Consider the heat capacity per unit mass C. For a mass element dm

$$dU = C dm\theta = C\rho\theta dV$$

so energy density $u = c\rho\theta$ and

$$\frac{\partial u}{\partial t} = c\rho\frac{\partial \theta}{\partial t} = k\nabla^2\theta$$

so

$$\frac{\partial \theta}{\partial t} = \frac{k}{c\rho}\nabla^2\theta$$

where $D = \frac{k}{c\rho}$ is the diffusivity.

5.2 Solution of 1D Wave Equation by Separation of Variables

Consider the problem of finding the displacement $u(x,t)$ of a plucked string. We have the initial conditions

$$u(x,0) = f(x)$$

$$\text{and} \;\; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

where $f(x)$ is the a triangular function with a peak at the point of plucking and two homogeneous boundary conditions

$$u(0,t) = u(L,t) = 0$$

where L is the length of the string. We anticipate a solution which is a linear combination of spacial and temporal functions

$$u(x,t) = \sum_n X_n(x)T_n(t)$$

5.2.1 Separating the Variables

Try $u(x,t) = X(x)T(t)$.

$$\frac{\partial^2}{\partial x^2}(XT) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(XT)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{v^2 T}\frac{\partial^2 T}{\partial t^2} = S$$

Where S is the separation constant, we are left with the following eigenvalue problems

$$\frac{d^2X}{dx^2} = SX \qquad\qquad \frac{d^2T}{\mathrm{d}t^2} = Sv^2T$$

5.2.2 Boundary Conditions

Using boundary conditions, solve eigenvalue problems to find the form of the solutions and eigenvalues and eigenfunctions (X_n, T_n). So using the homogeneous boundary conditions previously stated we can see

$$1:\ X(0) = 0$$
$$2:\ X(L) = 0$$

so we need to find solutions to

$$\frac{d^2X}{dx^2} = SX$$

which satifiy these BCs. Considering the first, three cases for the eigenvalues S: positive, zero or negative.

Case 1 - $S = k^2$ We have the differential equation

$$\frac{d^2X}{dx^2} = k^2X$$

Considering a solution

$$X = C_1e^{kx} + C_2e^{-kx}$$
$$\text{or } X = C_3\cosh(kx) + C_4\sinh(kx)$$

Then for BC 1, $C_3 = 0$. For BC 2 we see that

$$C_4\sinh(kL) = 0$$

so C_4 must also be zero. This gives us a solution of $X = 0$ which is trivial.

Case 2 - $S = 0$ The differential equation

$$\frac{d^2X}{dx^2} = 0$$

gives the solution $X = C_1x + C_2$. By comparing to boundary condition 1 we see that $C_2 = 0$ and then by boundary condition 2 $C_1 = 0$ - another trivial $X = 0$ solution.

Case 3 - $S = -k^2$ This time the differential equation is

$$\frac{d^2X}{dx^2} = -k^2X$$

which we know has a solution of the form

$$X = C_1\cos(kx) + C_2\sin(kx)$$

Applying the BCs to this solution we find that $C_1 = 0$ from BC 1 and for C_2:

$$C_2\sin(kL) = 0$$

hence kL must be an integer multiple of π. So we find that $k = k_n = n\pi/L$ where n is an integer and

$$X_n = C_2\sin\left(\frac{n\pi}{L}x\right)$$

We can also see that $S = -k^2 = -\frac{n^2\pi^2}{L^2}$ and we can find the time function $T(t)$ as

$$\frac{d^2T}{dt^2} = -\frac{n^2\pi^2v^2}{L^2}T$$

so we know

$$T_n = C_3\cos\left(\frac{n\pi v}{L}t\right) + C_3\sin\left(\frac{n\pi v}{L}t\right)$$

and the form of our full solution for $u(x,t)$ is

$$u_n(x,t) = X_nT_n = \sin\left(\frac{n\pi}{L}x\right)\left[a_n\cos\left(\frac{n\pi v}{L}t\right) + b_n\sin\left(\frac{n\pi v}{L}t\right)\right]$$

5.2.3 General Solution

We now know the form of $u(x,t)$ from analysing the eigenfunction problems produced by the initial separation of variables, but we still need to compare $u(x,t)$ with the initial conditions in order to find values for the constants a_n and b_n. Using the initial condition

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$

$$\left.\frac{\partial \sum u_n}{\partial t}\right|_{t=0} = \sum_n \frac{n\pi v}{L} \sin\left(\frac{n\pi}{L}x\right) \left[b_n \cos\left(\frac{n\pi v}{L}t\right) - a_n \sin\left(\frac{n\pi v}{L}t\right)\right] = 0$$

hence $b_n = 0$. We find a_n by matching the other initial condition $u(x,0) = f(x)$

$$\sum_n a_n \sin\left(\frac{n\pi}{L}x\right) \cos(0) = \sum_n a_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

where $f(x)$ is the form of the string at $t = 0$. We can regonise the form of this equation as that of a Fourier series, where the a_n are to be found by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Summary To solve a PDE by separation of variables:

1. Set up boundary conditions and initial conditions.
2. Apply separation of variables to reduce the PDE to ODE eigenvalue problems.
3. Apply homogenous boundary conditions to determine the form or the eigenvalues and eigenfunctions of the separated equations, with S the separation constant taking the values of $\pm k^2, 0$.
4. Apply initial conditions and any remaining boundary conditions to determine the coefficients of the general solution.

5.3 Laplace's Equation in Two Dimensions

In Cartesian co-ordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

For example, a square with side length L, with potential = 0 on three sides and $f(x)$ on the fourth. So we have the following boundary conditions:

$$\begin{aligned} 1:&\ \phi(x,0) = 0 \\ 2:&\ \phi(0,y) = 0 \\ 3:&\ \phi(L,y) = 0 \\ 4:&\ \phi(x,L) = f(x) \end{aligned}$$

5.3.1 Separting the Variables

Assume $\phi(x,y) = X(x)Y(y)$ so we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = S$$

5.3.2 Using Boundary Conditions

to find the general form of the solution, we have

$$\begin{aligned} \phi(0,y) = X(0)Y(0) = 0 \ &\Rightarrow\ X(0) = 0 \\ \phi(L,y) = X(L)Y(0) = 0 \ &\Rightarrow\ X(L) = 0 \end{aligned}$$

Case 1 $S = k^2$

$$\frac{d^2X}{dx^2} = k^2X$$

So our general solution would be of the form

$$X = C_1 \cosh kx + C_2 \sinh kx$$

BC 2 gives $C_1 = 0$, BC 3 gives $C_2 = 0$. So this is a trivial solution and is not the one we will use.

Case 2 $S = 0$

$$\frac{d^2X}{dx^2} = 0$$

So the general solution would be

$$X = C_1x + C_2$$

BCs give $X = 0$, again trivial.

Case 3 $S = -k^2$

$$\frac{d^2X}{dx^2} = -k^2 X$$

So the GS is

$$X = C_1 \cos kx + C_2 \sin kx$$

BC 2 gives $C_1 = 0$ and BC 3 gives $C_2 \sin kL = 0$, so $k = n\pi/L$ and our eigenfunction is

$$X = C_2 \sin\left(\frac{n\pi}{L}x\right)$$

Subbing k back into

$$\frac{d^2Y}{dy^2} = \frac{n^2\pi^2}{L^2}Y$$

we find

$$Y = C_3 \cosh\left(\frac{n\pi}{L}y\right) + C_4 \sinh\left(\frac{n\pi}{L}y\right)$$

and BC 1 tells us that $Y(0) = 0$, so $C_3 = 0$ and we can write the form of general solution (still with undetermined coefficients)

$$\phi_n(x, y) = a_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}y\right)$$

5.3.3 Determine the Coefficient a_n

The general solution is

$$\phi(x, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}y\right)$$

Boundary condition 4 tells us $\phi(x, L) = f(x)$ hence

$$\phi(x, L) = \sum_{n=1}^{\infty} a_n \sinh(n\pi) \sin\left(\frac{n\pi}{L}x\right)$$

Now if we let $b_n = a_n \sinh(n\pi)$ we want to find b_n such that

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

This is form of a sine only Fourier series, and hence b_n is the fourier coefficient

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

So the general solutions is

$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{b_n}{\sinh(n\pi)} \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}y\right)$$

What about inhomogeneous BCs? We can apply a superposition of the x and y solutions as the partial differential equation is linear.

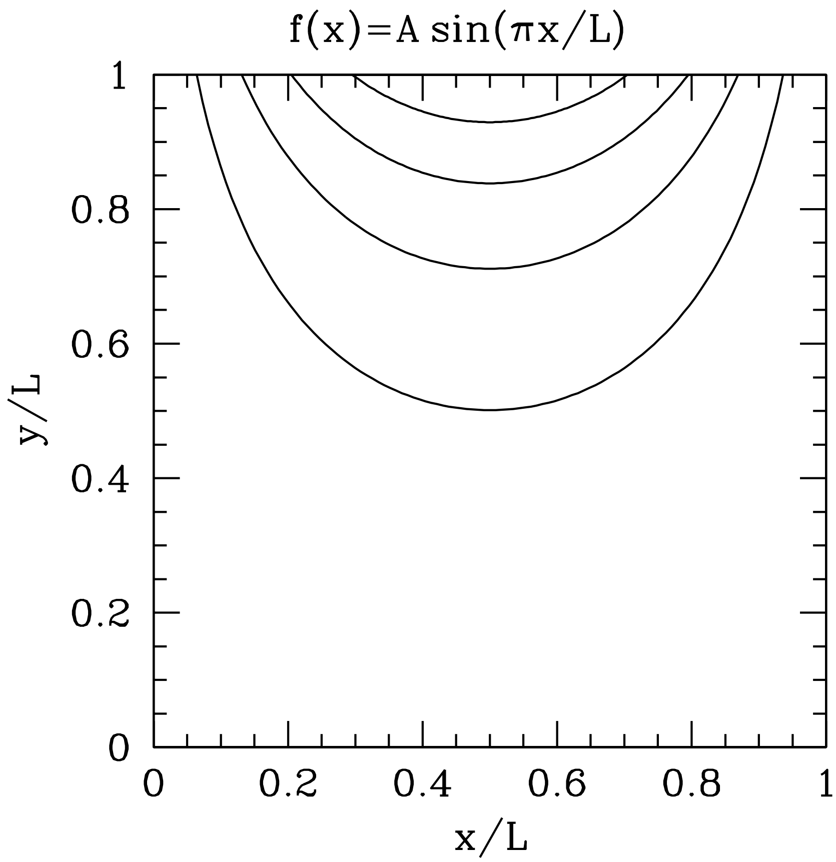

Figure 5.1: A plot of solutions to Laplace's equation for $f(x) = A \sin(\pi x/L)$.

6 Partial Differential Equations in Cyclindrical, Polar and Spherical Co-ordinates

The Laplacian can be transformed to cyclindrical, polar and spherical co-ordinate systems,

$$\text{Cylindrical} \quad \nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{6.1}$$

$$\text{Polar} \quad \nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} \tag{6.2}$$

$$\text{Spherical} \quad \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 u}{\partial \phi^2}\right) \tag{6.3}$$

where ρ is defined seperately from r as ρ is confined to the azimuthal angle $\phi = 0$ in circular and cylindrical co-ordinates and r is not confined this way in spherical co-ordinates; θ is the polar angle.

To express the Laplacian in different co-ordinate systems there are two methods - vector calculus or change of variables.

6.1 Laplace's Equation In Spherical Co-ordinates

We have the differential equation

$$\nabla^2 u = 0$$

e.g. steady state ($\partial_t u = 0$) temperature distribution throughout a conducting sphere, with spherical temperature distribution on the surface of the sphere.

Laplace's equation in spheical polars (6.3) can be solved by separation of variables, assuming that

$$u(r,\theta,\phi) = R(r)T(\theta)F(\phi)$$

and we have the boundary conditions

1: $F(\phi + 2n\pi) - F(\phi) = 0$
2: $T(\theta)$ is finite everywhere
3: $R(r)$ is finite everywhere
4: $u(a,\theta,\phi)$ - a is the surface of a sphere

So if we insert $u = RTF$ into (6.3) we find

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{\sin\theta}{T}\frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right) + \frac{1}{F}\frac{d^2 F}{d\phi^2} = 0$$

The last term is dependent only on ϕ hence we can say

$$\frac{d^2F}{d\phi^2} = SF$$

and we can try $S = 0, \pm m^2$. Using boundary condition 1 too:

Case 1 - $S = m^2$

No periodic solution.

Case 2 - $S = 0$

Constant F is the only periodic solution.

Case 3 - $S = -m^2$

Trying a solution of the form

$$F = A\sin(m\phi) + B\cos(m\phi)$$

so we know from the boundary condition that m must be an integer. So now we know that

$$\frac{1}{F}\frac{d^2F}{d\phi^2} = -m^2$$

Subbing this back into (6.3) and diving the whole equation by $\sin\theta$ we find

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0$$

Which is now in two distinct parts, one dependent on r alone and the other on θ.

If we set the θ dependent side equal to $-k$ it gives

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \left(k - \frac{m^2}{\sin^2\theta}\right)T = 0$$

and with a change of variable $x = \cos\theta$ and $X(x) = T(\theta)$

$$(1-x^2)\frac{d^2X}{dx^2} - 2x\frac{dX}{dx} + \left(k - \frac{m^2}{(1-x^2)}\right)X = 0$$

This is an associated legendre equation. Under the change of variable, BC 2 now says $-1 \leq x \leq 1$ and $k = l(l+1)$, where $l = 0, 1, 2, 3\ldots.$ For $m = 0$ the eigenfunctions are Legendre polynomials $P_l(x)$, for $m \neq 0$ the eigenfunctions are the associated Legendre polynomials $P_l^m(x)$. We now have a grid of integer values for l and m and the associated eigenfunctions. The combination of $T(\theta)F(\phi)$ gives *Laplace's spherical harmonics*:

$$P_l^m(x)\sin(m\phi) \quad \text{and} \quad P_l^m(x)\cos(m\phi)$$

Finally the r dependent eigenvalue equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = kR = l(l+1)R$$

is known as the Euler equation. The solution is of the form

$$R = c_1 r^l + c_2 r^{-l-1}$$

Boundary condition 1 tells us that $R(r)$ is finite for all r, hence $c_2 = 0$. So we have a general solution

$$u(r,\theta,\phi) = \sum_{l,m} r^l P_l^m(x)\left(A_l^m \sin(m\phi) + B_l^m \cos(m\phi)\right)$$

where $x = \cos\theta$ and we obtain constants by matching to BC 4.

Example Steady rate temperature distribution in a conduction sphere of radius a with BC

$$u(r = a) = 300 + 30x^2$$

where $x = \cos\theta$. What is $u(r,\theta,\phi)$?

Independent of ϕ, $m = 0$ is the only azimuthal eigenvalue so the GS reduces to

$$u(r,\theta) = \sum_l c_l r^l P_l(x)$$

We match the boundary condition at $r = a$ to determine c_l.

$$u(a,\theta) = \sum_l c_l' P_l(x) = 300 + 30x^2$$

So we must expand $300 + 30x^2$ in Legendre polynomials,

$$c_l' = \frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)dx$$

so for $l = 0$,

$$c_0' = c_0 a^0 = 310$$

for $l = 1$,

$$c_1 = 0$$

and for $l = 2$,

$$c_2' = c_2 a^2 = 20$$

So our complete solution becomes

$$u(r,\theta) = \sum_l c_l P_l(x) = 310 + 20\frac{r^2}{a^2} \cdot \frac{1}{2}(3x^2 - 1)$$

or

$$u(r,\theta) = 310 + 10\frac{r^2}{a^2}(3\cos^2\theta + 1)$$

6.2 The Wave Equation in Polar Co-ordinates

Recall Bessel's equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

which we solved using the Frobenius method. The equation leads to the individual equation $r^2 = p^2$ leaving two series solutions $J_{\pm p}$. Bessel functions are a decaying sine or cosine functions. Also recall that in the Frobenius method, problems arise when the idicial equation produces equal roots or roots separated by an integer.

For $p = 0$, $r^2 = 0$ so we have equal roots. The first solution is

$$y_1(x) = J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$

and it crosses the x-axis at the points $\alpha_1 = 2.405, \alpha_2 = 5.520, \alpha_3 = 8.654$. The second solution is not a power series and of form

$$y_2(x) = J_0(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{2n}$$

and it diverges at $x = 0$. The GS is

$$y = C_1 y_1 + C_2 y_2$$

6.2.1 In PDEs

Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + k^2x^2y = 0 \tag{6.4}$$

where k^2 is the constant of separation. We know that the Bessel function $J_p(z)$ is a solution of the DE

$$z^2\frac{d^2y}{dx^2} + z\frac{dy}{dx} + (z^2 - p^2)y = 0$$

and by writing $z = kx$ we can show

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(k^2x^2 - p^2\right)y = 0$$

$J_0(kx)$ is a solution of (6.4). Considered solution of (6.4) and subject to boundary conditions:

1. $y(0)$ is finite.
2. $y(a) = 0$.
3. $y(x, 0) = f(x)$ - initial displacement.

Boundary condition 2 leads to eigenvalues, $k_1 a = \alpha_1, k_2 a = \alpha_2, k_3 a = \alpha_3$, where k_n are the eigenvalues and the eigenfunction are $J_0(k_n x)$. We decompose $f(x)$ into a Bessel function series to match boundary condition 3

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(k_n x)$$

where coefficients are obtained using orthogonality condition

$$\int_0^a x J_0(k_n x) J_0(k_m x) dx = 0$$

when $m \neq n$ and where x is the weight function:

$$\int x f(x) J_0(k_m x) dx = \int x \left[\sum_{n=1}^{\infty} A_n J_0(k_n x)\right] J_0(k_m x) dx$$

hence using the orthogonality condition

$$A_m = \frac{\int_0^a x f(x) J_0(k_m x) dx}{\int_0^a x J_0(k_m x) dx}$$

6.2.2 Radial Vibration on a Drum

Displacement satisfies the wave equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where these is no polar angular term in the laplacian as there is circular symmetry. Using the separation of variable $u = R(r)T(t)$

$$\frac{1}{v^2 T}\frac{d^2 T}{dt^2} = \frac{1}{rR}\frac{dR}{dr} + \frac{1}{R}\frac{d^2 R}{dr^2} = -k^2$$

as we want sine/cosine like solutions. We can see that R satisfies the equation

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} + k^2 r^2 R = 0$$

and we have the boundary conditions:

1. $u(a,t) = 0$
2. $u(0,t)$ is finite.
3. $u(r,0) = f(r)$ - specified initial displacement.
4. $\frac{\partial u}{\partial t}\Big|_{t=0} = 0.$

BCs 1 and 2 give eigenfunction and values $J_0(k_n r)$, $k_n a = \alpha_n$ for each eigenvalue, solution for T is

$$T = A_n \cos(k_n v t) + B_n \sin(k_n v t)$$

and from BC 4, all $B_n = 0$. The general solution is a linear combination

$$u = \sum_{n=1}^{\infty} A_n J_0(k_n r)\cos(k_n v t)$$

7 Sturm-Liouville

All the PDE problems we have encountered require the solution of Sturm-Liouville problems. A S-L problem is defined by an ODE equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0$$

where λ is a constant, $r(x)$ is the weight function and p, p', q and r are all continuous and real; $r(x) > 0$ on the interval $a \le x \le b$ with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

where α_i and β_i are constants. These can be re-expressed as

$$p(a) = 0 \text{ requiring } y(a) \text{ and } y'(a) \text{ finite}$$
$$p(b) = 0 \text{ requiring } y(b) \text{ and } y'(b) \text{ finite}$$

or if $p(a) = p(b)$, periodic boundary conditions

$$y(a) = y(b)$$
$$y'(a) = y'(b)$$

Sturm-Liouville problems have the following properties:

1. The differential equation has the form
$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0$$
2. The eigenfunctions are orthogonal w.r.t. the weight function
$$\int_a^b r(x)Y_n(x)Y_m(x)dx = \delta_{nm}$$
3. The eigenvalues are real.
4. The eigenfunctions are complete on the interval $[a, b]$.

7.1 Example - SHM Equation

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

The SHM equation can be recast into Sturm-Liouville form by

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) + \lambda y = 0$$

where $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, interval $[0, 1]$, eigenfunction $\sin\left(\frac{n\pi x}{L}\right)$ so

$$\int_0^1 \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \delta_{nm}$$

with eigenvalues $n\pi/L$.

7.2 Notes on Fourier Series

One of the steps in solving PDEs is to evaluate coefficients by matching boundary conditions. In general we have

$$f(x) = \sum_n a_n g_n(x)$$

where we are interested in a_n. Orthogonality state

$$\int_a^b r g_n(x) g_m(x) dx = \delta_{nm}$$

where r is the weight function.

$$\int_a^b r f(x) g_m dx = \int_a^b r \left[\sum_n a_n g_n \right] g_m dx$$

so

$$a_m = \frac{\int_a^b r f(x) g_m dx}{\int_a^b r g_m^2(x) dx}$$