

## 2<sup>nd</sup> Year Optics 2011-12

### Problem Sheet 1 – Refraction and Mirrors

The purpose of this problem sheet is to give you experience of making basic calculations concerning refraction, mirrors and lenses.

1. A fish appears to be 2m below the surface of a pond filled with water of refractive index 1.33. What is the actual depth of the fish? You may assume that the angle of observation is small, i.e. that we stand above the fish and look down on it (it is helpful to consider a narrow cone of rays emerging from the fish and how they refract at the surface). [2.66m]
2. A ray of light is incident at a small angle  $\theta$  on a block of glass (refractive index  $n$ ) having thickness  $d$ . Derive an expression for the transverse displacement of the ray after it passes through the block of glass.
3. Find the critical angle for a water/air boundary. If a point below the surface emits light equally in all directions, what fraction of the light emitted will emerge from the surface of the water? [You will need to use the fact that the solid angle contained within a cone of half-angle  $\theta$  is  $2\pi(1 - \cos\theta)$ ] [17%]
4. Why does a normal camera lens not work as expected when immersed in water? [Hint: think about the focal length of the lens.] What condition is necessary if a camera lens is to work well both in and out of water?
5. An optical fibre is made of a glass core with a refractive index of 1.48 and cladding of refractive index 1.46.
  - a. Calculate the critical angle for this combination of materials.
  - b. Hence calculate the maximum angle of inclination to the axis of the fibre for light to propagate along the fibre without loss. [9.4°]
  - c. Calculate the number of times per m that light at this angle reflects off the wall of the fibre, assuming a diameter of 100 $\mu$ m.
  - d. What is the divergence angle of light emerging from the fibre into the air?
  - e. Calculate the NA of the fibre. [0.24]
6. For a concave mirror of radius of curvature 60cm find the position (or the apparent position) of the image and the lateral magnification when an upright object is located the following distances from the mirror. State whether the image is upright or inverted, and real or virtual:  
 (a) 60cm    (b) 45cm    (c) 30cm    (d) 15cm    (e) -15cm  
 Draw ray diagrams for each situation to confirm your calculation.
7. Repeat question 6 for a convex mirror of radius of curvature 60cm.

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### Problem Sheet 2 – Lenses

The purpose of this problem sheet is to give you experience of analysing optical systems comprising lenses

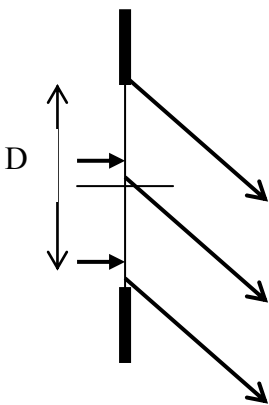
1. A sphere can act like a crude lens. Find the value of the refractive index necessary for parallel light to be focussed exactly onto the opposite side of a sphere.
2. Imagine that the eye is made of material of refractive index 1.38. Most of the focussing of rays of light occurs as the light enters the cornea. If the diameter of the eye is 24mm, find the radius of curvature of the front surface of the cornea needed to form on the retina an image of an object at infinity. [6.6 mm]
3. Starting from the lens equation, find an expression for the minimum distance between an object and an image for a lens of focal length  $f$ . [4 $f$ ]
4. For a given distance between object and image, there are always 2 possible positions of the lens. Find an expression for these two positions, and find the two positions if the distance between object and image is 90cm when the focal length is 20cm. Is the image identical in these two cases? [30, 60 cm; no]
5. For a converging lens of focal length 10cm, find the position (or the apparent position) of the image when an upright object is located the following distances from the lens. State whether the image is upright or inverted, and real or virtual:  
(a) 20cm (b) 11cm (c) 10cm (d) 9cm (e) 5cm.  
Draw ray diagrams for (a) and (e) to confirm your calculation.
6. Repeat the question 5 for a diverging lens of focal length -10cm.
7. One design of a zoom lens consists of a converging lens of focal length  $f_1=12\text{cm}$  and a diverging lens of focal length  $f_2=-18\text{cm}$ . The separation of the lenses,  $d$ , is variable from 0 to 4cm. Using the formula derived in the lectures, find the range of effective focal lengths of the zoom lens.
8. Chromatic aberration arises from the difference of refractive index for different wavelengths.
  - a. For silicate crown glass,  $n_{\text{red}}=1.50$ , and  $n_{\text{blue}}=1.52$ . Consider a converging lens made of this material that has a focal length of 10cm for red light. What is its focal length for blue light? Remember that  $f$  is proportional to  $1/(n-1)$ .
  - b. Write down the power of the lens in dioptres for both wavelengths
  - c. For silicate flint glass,  $n_{\text{red}}=1.61$ , and  $n_{\text{blue}}=1.66$ . Imagine that a diverging lens is made of this material with focal length  $f$  in the red (remember that  $f$  is a negative number). Find the power of this lens in dioptres in both the red and the blue.
  - d. If  $f$  is chosen correctly, a combination of these two lenses placed next to each other can have the same focal length for both blue and red light. [i.e. the total power of the combination is the same for both wavelengths.] Find the required value of  $f$  and the resulting focal length of the combination of lenses. [-20.5cm, 19.6cm]
9. This problem is to demonstrate an extreme example of spherical aberration so you can see how it arises. First, consider a crystal ball made from glass of refractive index 1.5. Using results from the lectures, demonstrate that parallel light close to the optical axis (i.e. paraxial rays) are brought to a focus *outside* the crystal ball.  
Now consider an extreme ray that just touches the ball with an angle of incidence close to  $90^\circ$ . Show that this ray crosses the optical axis inside the ball. Extreme rays are therefore brought to a focus in a different place from paraxial rays. This is the definition of spherical aberration.

10. Polaroid has a transmission of about 60% for light polarised parallel to its axis and 0% for light polarised perpendicular to this. Find the transmission of the following arrangements when they are illuminated with unpolarised light:
- a. One sheet of Polaroid
  - b. Two sheets of Polaroid with their axes parallel [18%]
  - c. Two sheets of Polaroid with their axes perpendicular
  - d. Three sheets of Polaroid with their axes at  $0^\circ$ ,  $45^\circ$  and  $90^\circ$ . Note that the insertion of a third sheet of Polaroid actually *increases* the transmission! [2.7%]
11. A wave plate is to be made using the birefringent material calcite which has refractive indices  $n_o=1.665$ ,  $n_e=1.490$ . For a wavelength of 550nm, calculate the thickness of the calcite for a quarter and half wave plate and comment on the results. [785.7nm, 1571.4nm]

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### Problem Sheet 3

1. It was stated in the lecture notes that the subsidiary maxima of the far field diffraction pattern of a slit are not exactly at the mid-point between the minima. Find an expression for the position of the subsidiary maxima which demonstrates that this is indeed the case.
2. When considering the Young's slits experiment, it was stated in lectures that as the width of the slit  $S_0$  increases, the visibility (or contrast) in the interference pattern reduces. The fringes will disappear completely when the angular displacement of the centre of the fringe patterns due to the contributions from the two extremes of the slit corresponds to one fringe in the fringe pattern. Use this to find an expression for the limiting size of slit width  $a$ . Compare this with the limiting value calculated in lectures by considering diffraction at  $S_0$  and the required illumination of the two slits
3. Consider a diffraction grating having  $N$  slits with spacing  $d$  and a total width  $D=Nd$ . Write down the condition for a principal maximum to occur at angle  $\theta$  for light of wavelength  $\lambda$  in first order.



In order to find the angle of the first minimum next to a maximum, we need to find the condition for the light from different parts of the grating to exactly cancel each other out. One way of doing this is to consider the grating to be split into two parts. The diagram shows this. If the contributions from the two arrowed points cancel each other out (i.e. are exactly out of phase), then the same will happen for all pairs of points across the slit, so the total intensity at this particular angle will be zero.

Find the condition for the fields from the two arrowed points to cancel each other out, and hence show that the difference in angle between the principal maximum and the first zero is  $\delta\theta = \lambda / D \cos \theta$  (remember that the small angle approximation can be used for  $\delta\theta$ ).

Note that this method can also be applied to diffraction from a single slit – the grating “looks” like a single slit of width  $D \cos \theta$ .

4. A diffraction grating is illuminated by a plane wave incident at angle  $\theta_i$ . By considering the optical path difference between light transmitted by adjacent slits, show that in this case the grating equation is given by:

$$d(\sin \theta_m - \sin \theta_i) = m\lambda$$

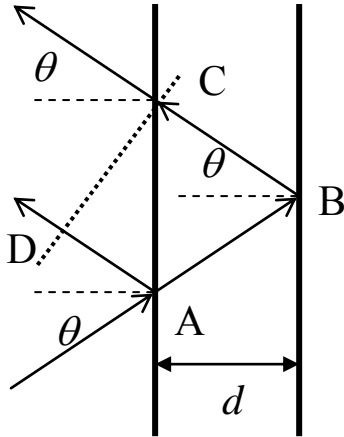
where  $m$  represents the order of the diffraction, and  $\theta_m$  is the angle at which the  $m^{\text{th}}$  order is observed.

5. A single mode optical fibre has a diameter of  $8 \mu\text{m}$ . Estimate the angular radius of the first minimum in the diffraction pattern of light of wavelength  $1.5 \mu\text{m}$  emerging from the fibre. (Hint: treat the fibre as a circular aperture.) [0.23 rad]

6. Find the magnitude of the angular diffraction at the iris of the eye, which is a circular aperture of diameter 3 mm. Given that the lens of the eye has a focal length of about 25mm, find the spread on the retina of the image of a point source. Consider two point objects at 100m separated by a small distance. How far apart do they need to be for their images to be resolved on the retina? (Use the Rayleigh criterion.) [22mm]
7. A grating spectrometer has a grating with 600 lines per mm and a width of 5cm. If it is mounted with a lens of focal length 2m, calculate the angles at which the first and second diffraction peaks occur. Find the width of the second order diffraction peak in the output plane for light of wavelength 633 nm. By considering the angle at which a second wavelength is just resolved from 633nm in the second order, calculate the wavelength which is just resolved, and hence the resolving power of the spectrometer. [77 $\mu$ m, 60000]
8. If a telescope has an objective of diameter 20mm, what is the limit on the angular resolution of the telescope (in  $\mu$ rad) due to diffraction at the entrance aperture for visible light? Noting that 60 arc seconds = 1 arc minute and 60 arc minutes = 1 degree, re-express this angular resolution in arc seconds. [6.8 arcsec]

## 2<sup>nd</sup> Year Optics 2011-2012 Problem Sheet 4

1. The path difference in a Michelson interferometer can be calculated using some trigonometry.



First find the path length between the two mirrors (i.e. the path ABC) in terms of  $d$  and  $\theta$  (assuming  $n=1$ ). The required path difference is not just the distance travelled between the mirrors, since the wavefront that goes through C intersects the other path at D, not at A. The path difference is therefore actually  $t = ABC - AD$ . The length AD can be found by trigonometry by first calculating AC. Calculate the value of  $t$  and show that it is given by

$$t = 2d \cos \theta$$

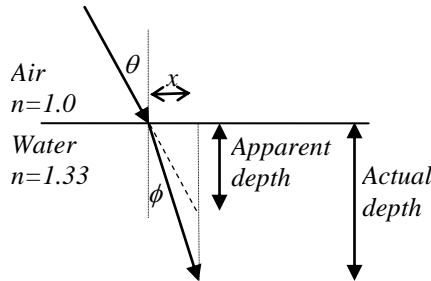
as quoted in the lectures. Note that it gets *smaller* as  $\theta$  increases, not larger as one might expect.

2. Consider a Michelson interferometer with the mirrors parallel, giving circular fringes. If the mirror spacing  $d$  is equal to approximately 1 mm and there is a bright fringe at  $\theta=0$ , find the value of  $\theta$  for the first and second bright fringes moving away from the centre of the pattern. Assume that  $\lambda=500\text{nm}$  and that  $\theta$  remains small so that a suitable expansion for  $\cos \theta$  can be used. If the fringe pattern is imaged onto a screen using a lens of focal length 10 cm, what are the radii of the first and second fringes? Find the values of these radii if  $d$  is increased to 1 cm. (Note that in order to do this question you do not need to know the order number precisely.) [0.7, 1.0mm]
3. What is the optical frequency of a spectral line of wavelength 546 nm? The line has a width due to the Doppler effect of around 2 GHz (which means a range of wavelengths are present). In a Michelson interferometer, the visibility of the interference fringes will reduce once the wavelength spread in the source gives a shift in the fringe pattern of one fringe. Calculate the value of  $d$  when the fringes start to lose visibility? Find this value also for the case of a stabilised laser with a linewidth of 1 MHz. [8cm, 150m]
4. What is the reflection coefficient for a substrate having a refractive index of 1.8? Find the ideal refractive index and thickness for an antireflection coating on glass with a refractive index of 1.8, for use at 600 nm. [0.08, 1.34, 112nm]
5. For the system described in Q4, consider what would happen for a wavelength of 300nm and 200nm. Would the coating still work for these wavelengths? (this requires some careful thought).
6. A laser illuminated Michelson interferometer is to be used for precision measurement. The laser has a wavelength of 633nm, and one of the mirrors of the interferometer is moved by a distance of 25m (remember this means that the optical path difference of the interferometer changes by 50m). This distance is to be measured by counting interference fringes. How many interference fringes are detected over this change in optical path difference? If the number of fringes is counted with an error of  $\pm 5$ , what is the error on the measurement of the distance?

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## Problem Sheet 1 – Solutions

1. Taking small angles:

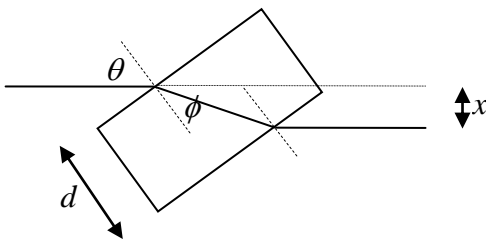


Apparent depth  $= x/\theta$   
 Actual depth  $= x/\phi$   
 Snell's law gives  $\phi = \theta/n$ .

So actual depth  $= nx/\theta = n \times$  apparent depth.

Given apparent depth of 2m and refractive index of 1.33,  
 So actual depth  $= 2 \times 1.33 = 2.66\text{m}$ .

2. From the geometry



$$x = \frac{d}{\cos \phi} \sin(\theta - \phi)$$

and

$$\sin \phi = \frac{\sin \theta}{n}$$

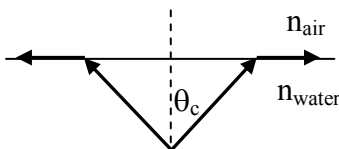
so

$$x = d \sin \theta \left[ 1 - \frac{\cos \theta}{n \cos \phi} \right]$$

For small angles this may be written

$$x = d\theta \left[ 1 - \frac{1}{n} \right]$$

3. The diagram shows the rays at the critical angle which refract along the boundary so do not escape.



$$\sin \theta_c = \frac{n_{\text{air}}}{n_{\text{water}}} = \frac{1}{1.33}$$

$$\theta_c = 49^\circ$$

Thus only light from the point source within a cone of half angle  $49^\circ$  will escape.  
 Solid angle of this cone is

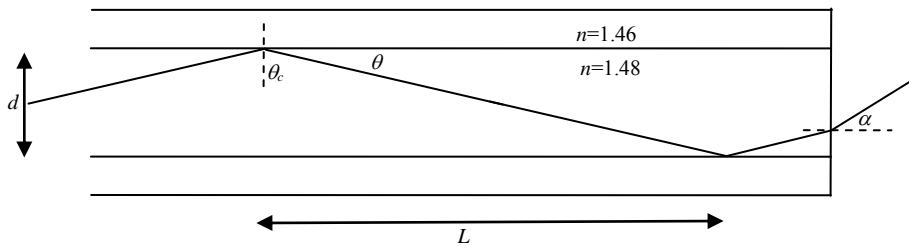
$$\Omega = 2\pi(1 - \cos 49^\circ) = 2\pi \times 0.34$$

Solid angle of a sphere is  $4\pi$ , so fraction escaping the surface is 17%.

4. The refraction at the front surface of the lens depends on the refractive index on both sides of the spherical surface (see the results derived in lectures). If one surface is water rather than air, it will not have as much focussing effect.  
 To have a lens that works well in water and air the refraction at the front surface has to be minimised. This happens for a distant object when the front surface is flat, such that

the incident rays are close to normal incidence. There would still be a difference in the lens' aberration in the two different situations (but that is not part of this course).

5.



- (a)  $n_{co}=1.48, n_{cl}=1.46$  therefore  $\sin(\theta_c) = 0.986, \theta_c = 80.6^\circ$ .
- (b) Maximum value of  $\theta = 90 - \theta_c = 9.4^\circ$ .
- (c)  $\tan(\theta) = \frac{d}{L}, \Rightarrow L = \frac{0.1}{0.166} = 0.6mm$ . This tells us that in a metre length of fibre this ray bounces 1660 times at this angle.
- (d) Using Snell's law at the exit of the fibre  $\sin(\alpha) = n \sin(\theta) = 0.242, \alpha = 14.0^\circ$ .
- (e) From definitions in lectures  $NA = n_o \sin(\alpha) = 0.242$ .

6.

For  $R=60cm, f=30cm$  (positive – concave mirror) using mirror equation gives

	$u$ (cm)	$v$ (cm)	Magnification	Comment
a	60	60	-1	Real image, inverted
b	45	90	-2	Real image, inverted
c	30	$\infty$	$\infty$	
d	15	-30	2	Virtual image, upright
e	-15	10	2/3	Real image, upright

The ray tracing diagrams, drawn to scale, are on a separate page.

Lines are coded as follows

Ray tracing rule 1: Long dash line

Ray tracing rule 2: Short dash line

Ray tracing rule 3: Solid line

Ray tracing rule 4: Long dash dot line.

In some cases not all 4 rules can be used.

7.

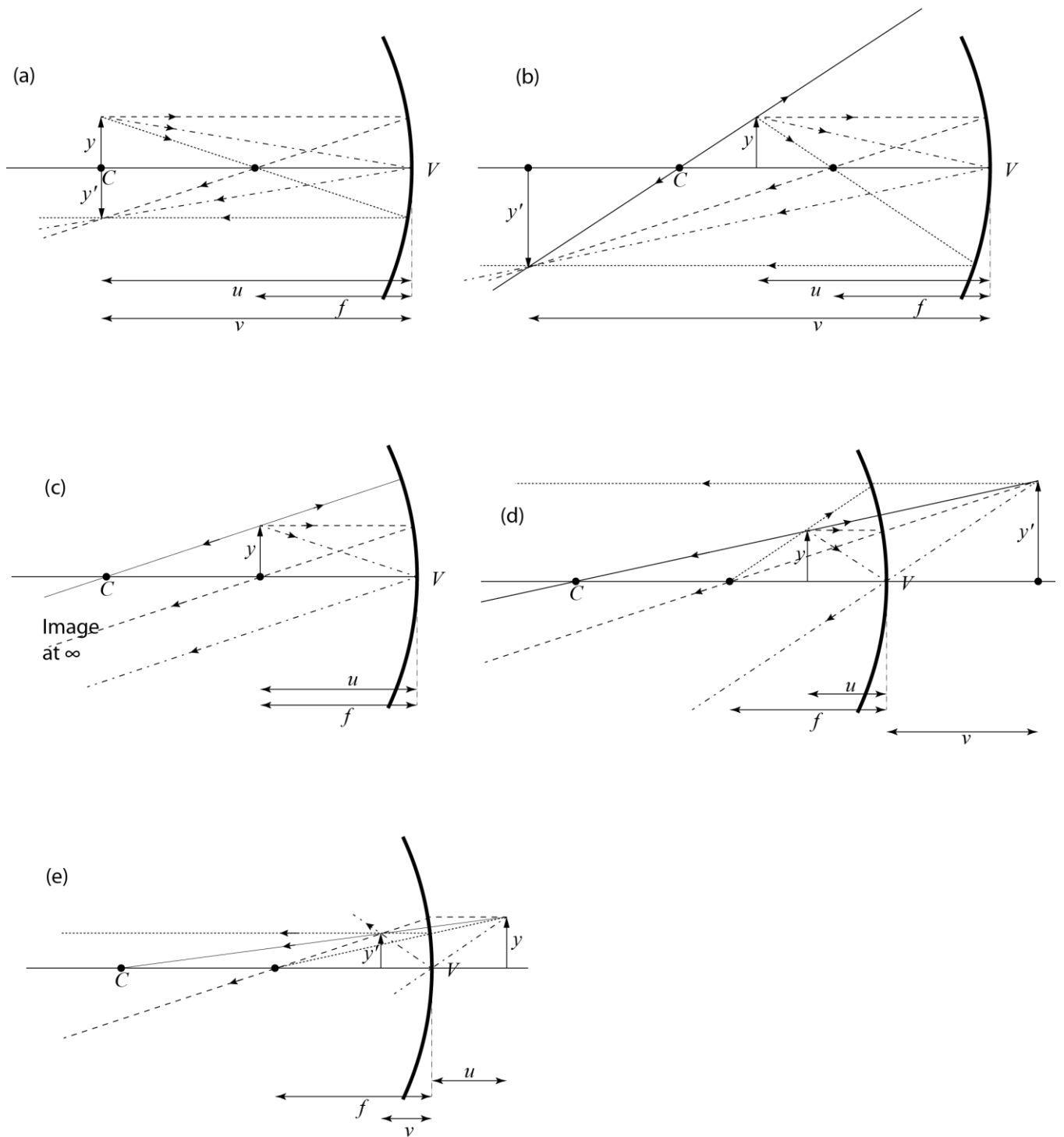
For  $R=-60cm, f=-30cm$  (negative – convex mirror) using mirror equation gives

	$u$ (cm)	$v$ (cm)	Magnification	Comment
a	60	-20	1/3	Virtual image, upright
b	45	-18	2/5	Virtual image, upright
c	30	-15	1/2	Virtual image, upright
d	15	-10	2/3	Virtual image, upright
e	-15	30	2	Real image, upright

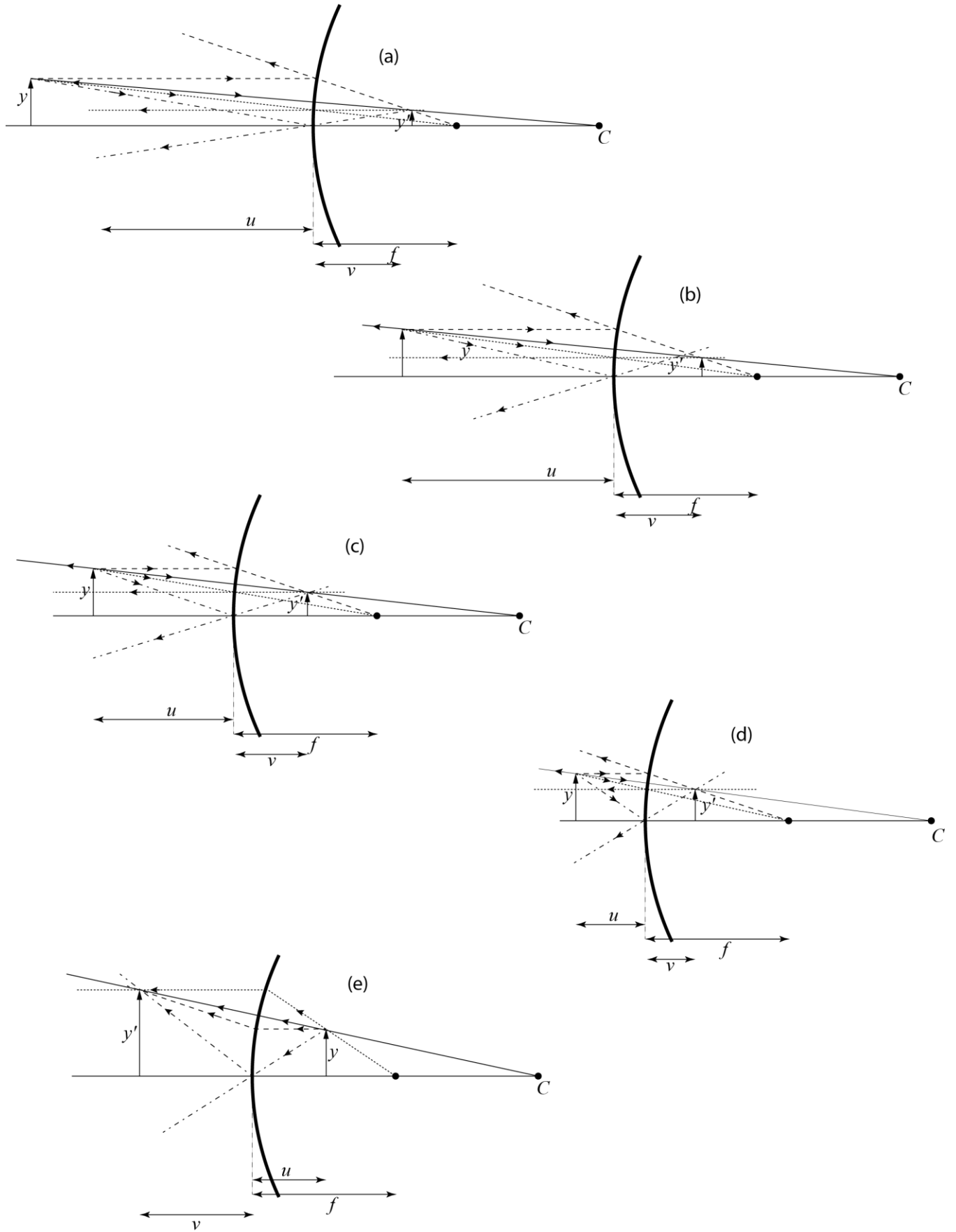
The ray tracing diagrams, drawn to scale, are on a separate page. Lines are coded as for question 6.



Ray tracing diagrams for problem sheet 1, question 6



Ray tracing diagrams for problem sheet 1, question 7



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### Problem Sheet 2 - solutions

1. For a single spherical surface

$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R}$$

We have  $u = \infty$  and require  $v = 2R$  if the rays are to be focussed at the back of the sphere, thus (taking  $n_1 = 1$ ):

$$\frac{n}{2R} = \frac{(n-1)}{R}$$
$$n = 2$$

This is only exactly true for rays close to the axis (paraxial rays). Rays further away from the axis will focus differently due to aberrations.

2. If all of the focusing is due to the front surface of the eye we need the equation for refraction at a single surface again

$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R}$$

Here  $u = \infty$  and require  $v = 24\text{mm}$ , with  $n_1 = 1$  and  $n_2 = 1.38$ :

$$R = \frac{1.38 - 1}{1.38} \times 24$$
$$= 6.6\text{mm}$$

3. Start with the lens equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}, \quad \text{so} \quad \frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{u - f}{uf}$$

set the object to image distance  $x = u + v$  then

$$x = u + \frac{uf}{(u - f)} = \frac{u^2}{(u - f)}$$

We want the minimum of  $x$  so differentiate and equate to zero

$$\frac{dx}{du} = \frac{(u - f)2u - u^2}{(u - f)^2} = \frac{u^2 - 2fu}{(u - f)^2} = 0$$

giving  $u = 0$  (not a physical result) and  $u = 2f$ , meaning  $v = 2f$  and minimum separation is  $4f$ .

4. From question 3 we have

$$x = \frac{u^2}{(u - f)}$$

which can be written

$$u^2 - ux + fx = 0$$

giving two possible values of  $u$

$$u = \frac{x}{2} \pm \sqrt{\frac{x^2}{4} - fx}$$

For  $x=90\text{cm}$ ,  $f=20\text{cm}$ ,  $u=45\pm 15$  the two cases are

- (i)  $u=60\text{cm}$ ,  $v=30\text{cm}$   $M=-0.5$  (image real, inverted and reduced)  
(ii)  $u=30\text{cm}$ ,  $v=60\text{cm}$   $M=-2$  (image real, inverted and magnified)

5. For  $f=10\text{cm}$  (Positive converging lens) using lens equation gives

	$u$ (cm)	$v$ (cm)	Magnification	Comment
a	20	20	-1	Real image, inverted
b	11	110	-10	Real image, inverted
c	10	$\infty$	n/a	
d	9	-90	10	Virtual image, upright
e	5	-10	2	Virtual image, upright

Note how the type of image changes as the object position moves across the focal plane.

The ray tracing diagrams, drawn to scale, are on a separate page.

Lines are coded as follows

Ray tracing rule 1: Long dash line

Ray tracing rule 2: Short dash line

Ray tracing rule 3: Solid line.

6. For  $f=-10\text{cm}$  (Negative diverging lens) using lens equation gives

	$u$ (cm)	$v$ (cm)	Magnification	Comment
a	20	-6.7	0.33	Virtual image, upright
b	11	-5.2	0.47	Virtual image, upright
c	10	-5	0.50	Virtual image, upright
d	9	-4.7	0.52	Virtual image, upright
e	5	-3.3	0.67	Virtual image, upright

The ray tracing diagrams, drawn to scale, are on a separate page.

Note all images are virtual.

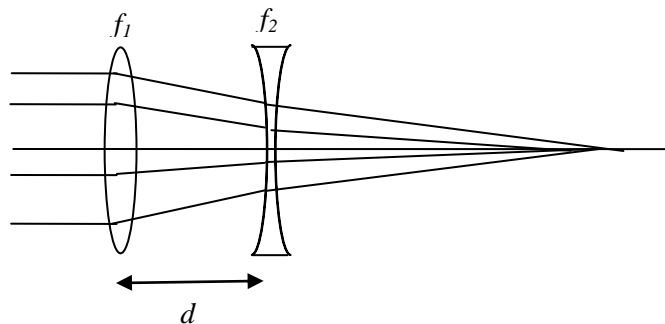
7.

$$f_1 = 12\text{cm}$$

$$f_2 = -18\text{cm}$$

$$d = 0 \text{ to } 4 \text{ cm}$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$



$$d = 0, \frac{1}{f} = \frac{1}{12} - \frac{1}{18} - \frac{0}{f_1 f_2} = \frac{1}{36}, f = 36\text{cm}$$

$$d = 4\text{cm}, \frac{1}{f} = \frac{1}{12} - \frac{1}{18} + \frac{4}{12 \times 18} = \frac{5}{108}, f = 21.6\text{cm}$$

8.

(a)  $f_1(\text{red}) = 10\text{cm}, f_1(\text{blue}) = 10 \times 0.5 / 0.52 = 9.6\text{cm}.$

(b) Power:  $P_1(\text{red}) = 10 \text{ dioptr}, P_1(\text{blue}) = 10.4 \text{ dioptr}.$

(c)  $f_2(\text{red}) = f, f_2(\text{blue}) = f \times 0.61 / 0.62 = 0.924f.$  Power:  $P_2(\text{red}) = 1/f \text{ dioptr}, P_2(\text{blue}) = 1.082/f \text{ dioptr}.$

(d) Condition required  $P_1(\text{red}) + P_2(\text{red}) = P_1(\text{blue}) + P_2(\text{blue})$

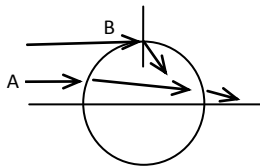
$$10 + \frac{1}{f} = 10.4 + \frac{1.082}{f}$$

$$\frac{0.082}{f} = -0.4$$

$$\frac{1}{f} = -4.9 \text{ dioptr}, f = -20.5\text{cm}$$

Final focal length (d=0 above)  $f = 19.6\text{cm},$  Power = 5.1 dioptr at both wavelengths.

9.



For the paraxial rays (e.g. A), the expression for refraction at a surface of radius of curvature  $R$  tells us  $\frac{n}{v} = (n-1)\left(\frac{1}{R}\right) \Rightarrow v = \frac{n}{(n-1)}R = 3R,$

hence the focus must be outside the sphere (after further refraction at the second surface).

The extreme ray (B) meets the sphere tangentially, and Snell's law tells us that the angle of refraction is  $42^\circ.$  As this angle is less than  $45^\circ$  the ray must cross the axis inside the sphere.

10.

a)  $T=30\%$  - incident light is unpolarised it can be considered as having 50% in two orthogonal polarisations.

b) Considering effect of each component in turn -  $T = 50 \times 0.6 \times 0.6 = 18\%$

c)  $T = 50 \times 0.6 \times 0.0$

d)  $T = 50 \times 0.6^3 \times \cos^2 45^\circ \times \cos^2 45^\circ = 2.7\%$  at each of polariser 2 and 3 Malus' law is used to determine the component of polarisation in the direction of the transmission axis of the polariser, and each polariser transmits 60%.

11.

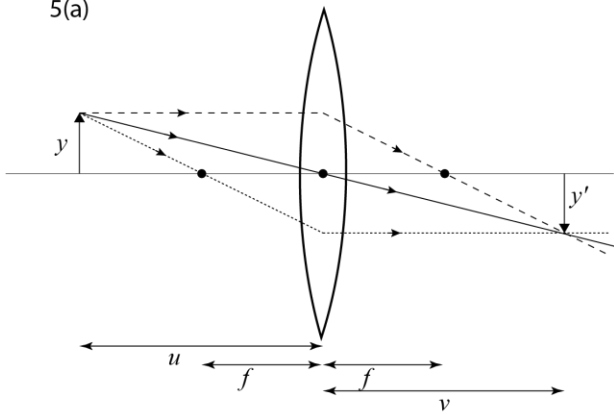
The phase difference introduced between the two orthogonal directions in the wave plate is given by

$$\phi = \frac{2\pi}{\lambda}(n_o - n_e)d$$

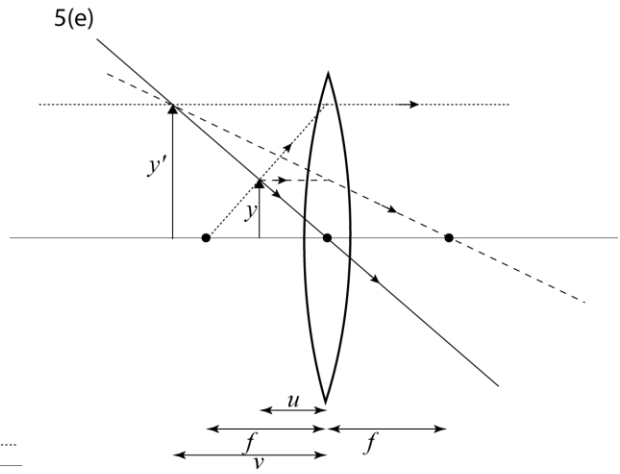
where  $d$  is the thickness of the plate. For a quarter wave plate  $\phi = \pi/2$  and substitution give  $d = 785.7\text{nm},$  for a half wave plate  $\phi = \pi$  and substitution give  $d = 1571.4\text{nm}.$  This means that the calcite plates are very thin and because of this the birefringent plates are often made "multiple order", where for example in an quarter wave plate the phase difference introduced is  $\phi = 2m\pi + \pi/2$  with  $m$  an integer.

Ray tracing diagrams for problem sheet 2, questions 5 & 6

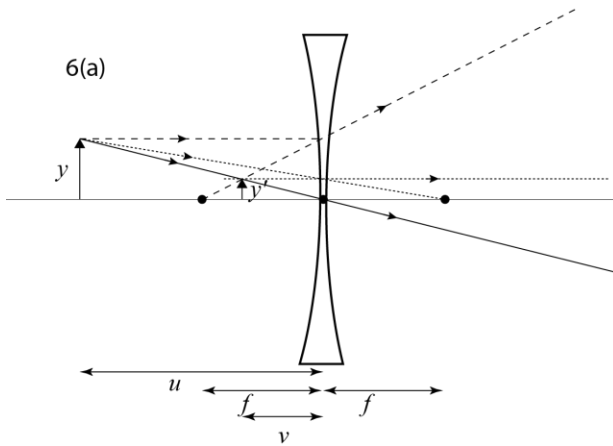
5(a)



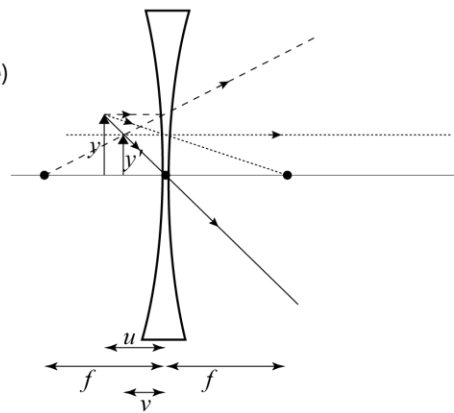
5(e)



6(a)



6(e)



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## Problem Sheet 3 – Solutions

(with numbering of solutions 3 and 4 corrected to match questions!)

1.

The diffraction pattern of a single slit is

$$I = I_o \frac{\sin^2 \beta}{\beta^2}$$

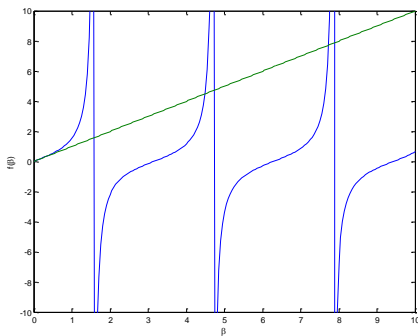
To find the maxima (and minima) differentiate with respect to  $\beta$  and equate to zero:

$$\frac{dI}{d\beta} = I_o \frac{\beta^2 2 \sin \beta \cos \beta - \beta \sin^2 \beta}{\beta^4} = I_o \frac{\sin \beta (\beta \cos \beta - \sin \beta)}{\beta^3} = 0$$

The two results which give zero in numerator are  $\sin \beta = 0$ , giving the position of the minima (which we could do by inspection of the original expression), or the expression in brackets which can be rewritten

$$\tan \beta = \beta$$

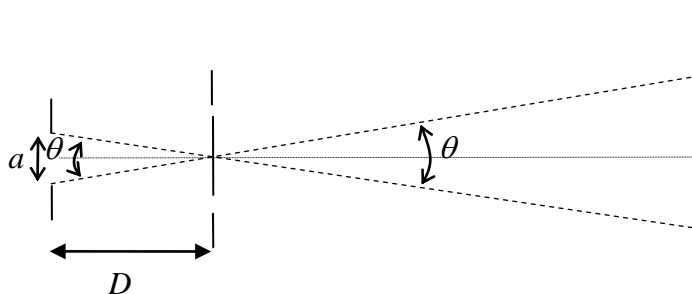
This transcendental equation can be solved graphically to demonstrate the point that the maxima are not exactly between the minima (but gets closer the further out the peaks are as shown in the figure which plots  $y = \tan \beta$ , and  $y = \beta$ )



2.

Angular displacement of one fringe in interference pattern for Young's slits of separation  $d$  is given by (small angle approx)

$$\theta = \frac{\lambda}{d}$$



From the geometry, we also have

$$\theta = \frac{a}{D}$$

Hence for visible fringes the requirement is

$$\theta = \frac{a}{D} < \frac{\lambda}{d},$$

$$a < \frac{\lambda D}{d}$$

Comparing with the diffraction calculation in lectures, this is the same order of magnitude.

3.

Max at angle  $\theta$  if  $d \sin(\theta) = \lambda$  ( $m=1$ ).

First minimum is at angle  $\theta + \delta\theta$ , and an addition  $\lambda/2$  path difference is needed across half

$$\frac{1}{2}Nd \sin(\theta) = N \frac{\lambda}{2}, \quad (\text{for maximum}) \quad \frac{1}{2}Nd \sin(\theta + \delta\theta) = N \frac{\lambda}{2} + \frac{\lambda}{2}, \quad (\text{for minimum}).$$

also the identity

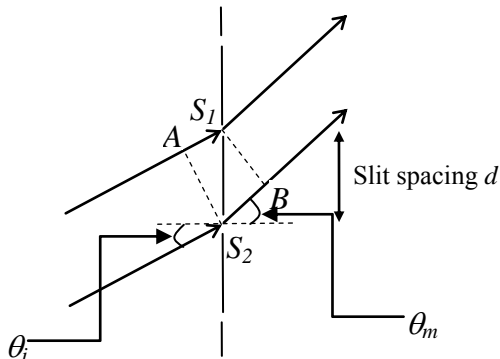
$$\sin(\theta + \delta\theta) = \sin \theta \cos \delta\theta + \cos \theta \sin \delta\theta \approx \cos \theta \delta\theta + \sin \theta$$

hence

$$\frac{1}{2}Nd \cos(\theta) \delta\theta = \frac{\lambda}{2}, \quad \delta\theta = \frac{\lambda}{Nd \cos(\theta)} = \frac{\lambda}{D \cos(\theta)}$$

4.

The optical path difference between adjacent slits is given by the difference between the paths  $S_2B$  and  $AS_1$ .



From the geometry

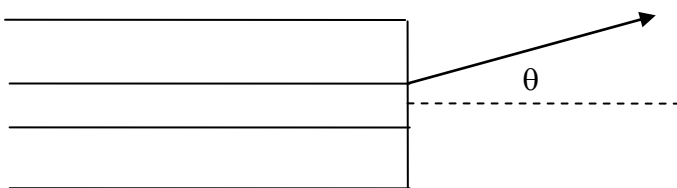
$$AS_1 = d \sin \theta_i$$

$$S_2B = d \sin \theta_m$$

For constructive interference in the  $m^{\text{th}}$  order we obtain the modified grating equation

$$d(\sin \theta_m - \sin \theta_i) = m\lambda.$$

5.



Approximate the core is a diffracting aperture  $\theta \approx 1.22 \frac{\lambda}{d} = 1.22 \frac{1.5}{8} = 0.23 \text{ rad} = 13^\circ$ . This is an approximation as the presence of the cladding modifies the output pattern, but it is a good estimate

6.

To calculate the angular radius of the Airy pattern for diffraction at the pupil requires an estimate of the wavelength of light. Using  $\lambda = 550 \text{ nm}$  (mid-visible) we get

$$\theta \approx 1.22 \frac{\lambda}{d} = 1.22 \frac{0.55}{3000} = 0.22 \text{ mrad}.$$

Corresponding radius on the retina (assuming  $f_{\text{eye}} = 25 \text{ mm}$ )  $r = 0.22 \times 10^{-3} \times 25 = 5.5 \mu\text{m}$ ,  $d = 11 \mu\text{m}$ .



Rayleigh criterion to resolve two point objects separated by a distance  $a$ , at a distance  $D$  ( $=100\text{m}$ ) is that their angular separation must be at least the same as the angular radius of the first minimum in the diffraction pattern. Hence angular separation

$$\theta = \frac{d}{D} = 0.22 \text{ mrad}, \text{ and } d = 22\text{mm}.$$

7.

Slit separation  $d = \frac{1}{\text{Lines per mm}} = \frac{1}{6000} = 1.67\mu\text{m}$ , with  $\lambda=633\text{nm}$ , and in the second order

( $m=2$ ) the angular position of the diffraction maximum,  $\theta$ , comes from

$$\sin \theta = \frac{m\lambda}{d} = \frac{2 \times 0.633}{1.67} = 0.76, \quad \theta = 0.860\text{rad} = 49.3^\circ.$$

Angular radius of this peak (to the first minimum,  $\delta\theta$ , from  $\delta\theta = \frac{\lambda}{D \cos \theta} = 19.5\mu\text{rad}$ . With a 2m lens this gives a peak (full) width  $d=2x f x 19 \times 10^{-6} = 77\mu\text{m}$ .

The wavelength which is just resolved (via Rayleigh) has its second order diffraction peak

at angle  $\theta + \delta\theta$  so  $\sin(\theta + \delta\theta) = \frac{2\lambda}{d}$ ,  $\lambda = \frac{0.758 \times 1.67}{2} = 0.63301\mu\text{m}$ .

$$\text{Resolving power } R = \frac{\lambda}{\Delta\lambda} = \frac{633}{0.01} \approx 60000$$

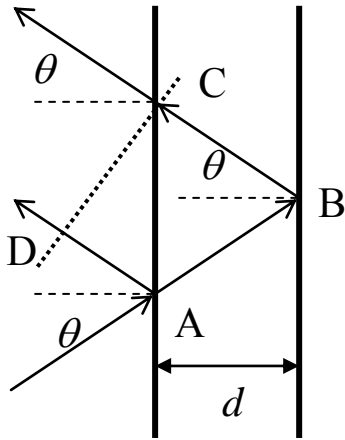
8.

Diameter of aperture  $D=20\text{mm}$ , angular resolution,  $\theta = 1.22 \frac{\lambda}{D} = 33\mu\text{rad}$  (in visible).

$$\theta = 33\mu\text{rad} = 0.0019^\circ = 0.113 \text{ arc minutes} = 6.8 \text{ arc seconds}.$$

## 2<sup>nd</sup> Year Optics 2011-12 Problem Sheet 4 – Solutions

1.



The path difference is  $t = ABC - AD$ .

$$AB = \frac{d}{\cos \theta} = BC$$

$$AC = 2d \tan \theta$$

$$\therefore AD = 2d \tan \theta \sin \theta$$

Hence

$$\begin{aligned} ABC - AD &= \frac{2d}{\cos \theta} - 2d \tan \theta \sin \theta \\ &= \frac{2d}{\cos \theta} [1 - \sin^2 \theta] \\ &= 2d \cos \theta \end{aligned}$$

In general optical path difference is multiplied by  $n$ , the refractive index (here  $n=1$ ).

2.

$d=1\text{mm}$ , bright fringe at  $\theta=0$ . Using result from lectures (derived in problem 1)

$$\left(m + \frac{1}{2}\right)\lambda = 2d, \quad n=1, \theta=0$$

next bright fringe moving away from centre at angle  $\theta_1$  where

$$\begin{aligned} \left(m - \frac{1}{2}\right)\lambda &= 2d \cos \theta_1 \\ &= 2d \left[1 - \frac{\theta_1^2}{2}\right] \end{aligned}$$

using small angle approximation for cos. Subtracting these equation:

$$\lambda = d\theta_1^2$$

$$\therefore \theta_1 = \sqrt{\frac{\lambda}{d}} = 0.022 \text{ rad} \quad \text{similarly} \quad \theta_2 = \sqrt{\frac{2\lambda}{d}} = 0.032 \text{ rad}$$

If focal length of the lens,  $f$ , is 10cm, the radius of this rings is given by  $f\theta$ . So,  $r_1=2.2\text{mm}$ ,  $r_2=3.2\text{mm}$ . For  $d=1\text{cm}$ , all are smaller by  $\sqrt{10}$ ,  $r_1=0.70\text{mm}$ ,  $r_2=1.0\text{mm}$

3.

Fringes will start to lose visibility when the fringe patterns are out of step by one fringe. This requirement may be written in terms of phase:

$$\frac{2\pi}{\lambda} 2d - \frac{2\pi}{(\lambda + \Delta\lambda)} 2d = 2\pi, \quad \rightarrow \quad \frac{1}{2d} = \left(\frac{\Delta\lambda}{\lambda(\lambda - \Delta\lambda)}\right) \approx \frac{\Delta\lambda}{\lambda^2}$$

or in terms of optical path difference

$$2d = m\lambda, \quad 2d = (m-1)(\lambda + \Delta\lambda)$$

and eliminating  $m$ ,

$$2d \approx \frac{\lambda^2}{\Delta\lambda}$$

Also from  $c = v\lambda$

$$\frac{\Delta\lambda}{\lambda^2} = -\frac{1}{\lambda} \frac{\Delta v_D}{v}$$

using values given  $\lambda = 546\text{nm}$ ,  $\rightarrow v = c/\lambda = 5.5 \times 10^{14} \text{Hz}$ ,  $\Delta v_D = 2 \times 10^9 \text{Hz}$  hence  $d = 7.5\text{cm}$ .  
For a stabilised laser with linewidth = 1MHz,  $d = 150\text{m}$ .

4.

Assume substrate with  $n=1.8$  is in air ( $n=1$ ) then at normal incident:

$$|r| = \left| \frac{1-1.8}{1+1.8} \right| = 0.29, \quad \Rightarrow \quad R = |r|^2 = 0.08 \text{ (8\%)}$$

For antireflection coating need refractive index of coating  $n = \sqrt{1.8} = 1.3$  (there is an optical material, cryolite, with  $n=1.36$ ). Film thickness has to be quarter wave which given wavelength of 600nm is ( $n=1.3$ ):

$$d = \frac{\lambda}{4n} = 115\text{nm}.$$

5.

For wavelength of 300nm, layer is not a quarter wave thick, and a thickness of 115nm corresponds to  $d = \frac{\lambda}{2n}$  which means the layer is a half wavelength thick. This means that the difference in path is  $\lambda$ , and the waves are in phase. There is constructive interference, as the waves are in phase. The amplitude of the total reflection still depends on the reflection at the boundaries of the thin film. Detailed calculation shows that the reflectance is 8% (the same as without the layer).

At 200nm, a 115nm layer is three-quarters of a wavelength thick, and waves are out of phase again and coating is antireflection again.

(In both cases dispersion has been ignored, and would change the exact wavelength at which these new conditions are satisfied).

6.

The mirror moves 25m, and every  $\lambda/2$  the output of the interferometer traces through a complete fringe. Therefore the total number of fringes is  $25/(633 \times 10^{-9}/2) = 78,988,941$  (ignoring the fractional part). An error of  $\pm 5$  fringes is a relative error of  $1 \times 10^{-7}$ , which corresponds to an error in the measurement of the distance of  $\pm 1.6 \mu\text{m}$  (or 5 fringes error means  $5 \times \lambda/2 = 1.6 \mu\text{m}$ ).