Imperial College 4th Year Physics UG, 2012-13

General Relativity

Revision lecture

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1 Exam

This is 2 hours. There is one compulsory question ('section A') (worth 40 marks) and then you have a choice of two of four ('section B'), (each worth 30 marks).

I regard the compulsory section as testing more basic things. The optional questions are intended to be harder and will often cover more advanced material or trickier calculations.

I usually structure questions so that the first parts are easier and they get progressively harder. So you should attempt all questions, particularly the beginning parts. If you run out of time and don't tackle a question you will be throwing away a lot of easy marks.

Since the section 'A' compulsory question tests more basic things, *and* is worth more marks than the optional questions, make sure you attempt all of it. Don't rush onto the optional questions.

Everyone who has revised and taken the course (done example sheets + attended lectures) should do well. However I expect no one will get 100%.

I expect you to know important formulae, but I will give you some important ones as in the Mock exam.

2 Overall structure of the course

The structure of the course is;

Basic geometry:

Tensors, metrics, LIF, geodesics (and E-L), curvature.

- Chapter 2: Riemannian geometry
- Chapter 6: Curved spacetime and a geometric origin to gravity
- Chapter 7: Curvature

This is all really important to know well.

Example sheets: 3, 4 and 7. Also sheets 1 and 2.

Special relativity and physics in curved spacetime:

Special relativity as physics in Minkowski space-time. Laws of motion as tensor expressions - do not depend on our choice of coordinates. Matter and stress tensors. Perfect fluids.

- Chapter 4: The geometry of Special Relativity
- Chapter 5: Continuous matter in Minkowski space-time

This is more conceptual and where the physics is introduced; there isn't that much to 'know'.

Example sheets: 5

Einstein equation and the Newtonian space-time:

- Chapter 6: Curved spacetime and a geometric origin to gravity
- Chapter 8: Einstein's equations

Understanding how space-time curvature can give the illusion of 'Newtonian gravity' is crucial to understand. Understanding Einstein's equations is crucial.

Example sheets: 6 and 8.

More advanced: Cosmology and Black holes

- Chapter 9: The FLRW space-time and cosmology
- Chapter 10: The Schwarzschild space-time

These are more advanced topics. The important thing is to understand how we compute geodesics, curvature, Einstein equations. I wouldn't ask you to reproduce this without help, but you must understand these conceptually.

Example sheet: 8

3 The example sheets

I have tried to make the example sheets useful to aid in understanding what you learn in lectures and to give you practice at manipulating tensors and geometry. It is crucial that you do practice, as it is very hard to really understand what is going on unless you get your 'hands dirty'.

The questions are not structured like exam questions - for that, see the mock exam. However, the types of things that I ask you to do in the rapid feedback questions are exactly the sort of thing you might be asked in the exam. (However in the exam the question would be broken into parts, which will help to guide you). You can certainly expect that parts of calculations done in the example sheets may well comprise parts of exam questions.

The 'not for rapid feedback questions' are usually much longer. Obviously since the exam is only 2 hours I'm not going to set anything like such long questions. However these long questions are useful to really gain practice with the types of calculation in GR so it is a good idea to try them. Also, small parts of those calculations might indeed be the sort of thing I ask in the exam.

4 A (probably incomplete) summary of some of the key ideas

4.1 Geometry; Chapter 2, 6 and 7

Take coordinates x^i. Under a change of coordinates, $x^i \to x'^{i'} = x'^{i'}(x)$, we define a transformation matrix, $\mathbf{M}$, with components,

$$M^{i'}{}_{j} \equiv \frac{\partial x'^{i'}(x)}{\partial x^j} \tag{1}$$

Note that the inverse of this matrix is,

$$M^{i}{}_{j'} \equiv \frac{\partial x^{i}(x)}{\partial x'^{j'}} \tag{2}$$

so that,

$$M^{i'}{}_{j} M^{j}{}_{k'} = \delta^{i'}_{k'} \quad \text{and} \quad M^{i}{}_{j'} M^{j'}{}_{k} = \delta^{i}_{k} \tag{3}$$

A **scalar** or **function** transforms trivially,

$$f'(x') = f(x(x')) \tag{4}$$

A **vector** $v^i(x)$ has the property that it transforms as,

$$v'^{i'}(x') = M^{i'}{}_{j}(x) v^j(x) \tag{5}$$

A **covector**

$$w'_{i'}(x') = M^{j}{}_{i'}(x) w_j(x) \tag{6}$$

A (q, r) **tensor** has q 'up' indices and r 'down' indices. Under a transform to new coordinates $x^{i'}$ then the new components are,

$$T'^{i'_1 i'_2 \ldots i'_q}{}_{j'_1 j'_2 \ldots j'_r}(x') = M^{i'_1}{}_{i_1} \ldots M^{i'_q}{}_{i_q} M^{j_1}{}_{j'_1} \ldots M^{j_r}{}_{j'_r} T^{i_1 i_2 \ldots i_q}{}_{j_1 j_2 \ldots j_r}\Big|_{x(x')} \tag{7}$$

We usually define the components of a tensor in a particular coordinate system; eg. Minkowski metric in Minkowski coordinates $x^\mu = (t, x^i) = (t, x, y, z)$ is,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

or alternatively,

$$g_{tt} = -1 \ , \quad g_{ti} = g_{it} = 0 \ , \quad g_{ij} = \delta_{ij} \tag{9}$$

These are not tensor equations - they are only true in the Minkowski coordinates. However when we write equations such as;

$$T_{\mu\nu} = \rho \, v_\mu v_\nu \tag{10}$$

we mean this as a tensor expression, holding in all coordinates, with $T_{\mu\nu}$, ρ and v^μ themselves being tensors.

The line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{11}$$

tells us the infinitesimal space-time interval, ds^2 between two points separated by the infinitesimal coordinate displacement dx^μ.

If $ds^2 > 0$ the interval is space like and ds is the infinitesimal proper distance. If $ds^2 < 0$ then the interval is time like and the infinitesimal proper time, $d\tau$, is given by $d\tau^2 = -ds^2$. An interval with $ds^2 = 0$ is null/light-like.

Since ds^2 is invariant under coordinate transformations it defines the metric tensor, $g_{\mu\nu}$. The metric is a *symmetric* $(0, 2)$ tensor; $g_{\mu\nu} = g_{\nu\mu}$. It must be invertible, and the inverse is a $(2, 0)$ tensor $g^{\mu\nu}$ so that,

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \tag{12}$$

The metric determines the space-time line element.

For a Lorentzian metric, at every point the metric is a symmetric matrix with 1 negative and 3 positive eigenvalues, corresponding to the time direction and 3 space directions.

LIF coordinates; For a Lorentzian metric we may always choose co-ordinates so that at some chosen point $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$ (so that ${\Gamma^\mu}_{\alpha\beta} = 0$), ie. at some point x_p,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + O((x - x_p)^2) \tag{13}$$

A curve is described parametrically, $x^i = x^i(\lambda)$ in terms of a parameter λ. The tangent vector is $v^i = dx^i(\lambda)/d\lambda$. Recall this correctly transforms as a vector (due to the chain rule).

Timelike curves (the trajectories of massive particles) have $v^\mu v^\nu g_{\mu\nu} < 0$. For spacelike curves $v^\mu v^\nu g_{\mu\nu} > 0$ and for null (trajectories of light rays) $v^\mu v^\nu g_{\mu\nu} = 0$.

We measure the proper distance s along a space like curve by integrating the infinitesimal proper distance along each $d\lambda$ as,

$$s = \int d\lambda \sqrt{g_{ij} v^i v^j} \tag{14}$$

We measure the proper time τ along a time like curve by integrating the infinitesimal proper time along each $d\lambda$ as,

$$\tau = \int d\lambda \sqrt{-g_{ij} v^i v^j} \tag{15}$$

The metric raises/lowers indices;

$$T^{i_1\ldots i_{n-1}}{}_a{}^{i_{n+1}\ldots i_q}{}_{j_1\ldots j_r} = g_{a\,i_n} T^{i_1\ldots i_q}{}_{j_1\ldots j_r} \qquad (16)$$

Under an infinitesimal active shift of our coordinates ('infinitessimal diffeomorphism'); $x^i \to x'^i = x^i - \epsilon v^i$, we have,

$$\begin{aligned} f(x) \to f'(x) &= f(x) + \epsilon \mathrm{Lie}(v, f)\ , \quad \mathrm{Lie}(v, f) \equiv v^i \partial_i f \\ w^i(x) \to w'^i(x) &= w^i(x) + \epsilon(\mathrm{Lie}(v, w))^i\ , \quad (\mathrm{Lie}(v, w))^j \equiv v^i \partial_i w^j - w^i \partial_i v^j \end{aligned}$$

In general a tensor shifts by its Lie derivative with respect to v for an infinitesimal diffeomorphism.

Given a tensor (q, r) tensor, ω, we say that if for some vector field v^i we have $\mathrm{Lie}(v, \omega) = 0$ then the diffeomorphism generated by v is a symmetry of the tensor ω.

For the metric;

$$\mathrm{Lie}(v, g)_{ij} = \left(v^k \partial_k g_{ij} + g_{ik} \partial_j v^k + g_{jk} \partial_i v^k\right) \qquad (17)$$

[Note that for the metric (and only the metric) this Lie derivative can actually be written in terms of the covariant derivative; $\nabla_{(i} v_{j)} = 0$.]

For the Lie derivative of a general tensor see the notes.

Example. Consider a vector field v^i and a tensor $T^{i_1\ldots}{}_{j_1\ldots}$. Take coordinates so that the vector field $v^i = (1, 0, \ldots, 0)$. Then if v is a symmetry of T then,

$$Lie(v, T)^{i_1\ldots}{}_{j_1\ldots} = v^m \frac{\partial}{\partial x^m} T^{i_1\ldots}{}_{j_1\ldots} = 0 \qquad (18)$$

so that the components $T^{i_1\ldots}{}_{j_1\ldots}$ do not depend on the coordinate x^1.

A *symmetry* of the metric is called an *isometry*. If for some vector v^i we have $\mathrm{Lie}(v, g) = 0$ (so equivalently $\nabla_{(i} v_{j)} = 0$) we say v^i generates an isometry of the metric. In fact then v^i is called a 'Killing' vector. Ex. Schwarzschild space-time - t is Killing vector.

The Christoffel symbol (not a tensor) is defined as;

$$\Gamma^c{}_{ab}(x) \equiv \frac{1}{2} g^{cd} \left(\frac{\partial g_{db}}{\partial x^a} + \frac{\partial g_{da}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d} \right) \tag{19}$$

It has symmetry;

$$\Gamma^c{}_{ab} = \Gamma^c{}_{ba} \tag{20}$$

It determines the first partial derivatives of the metric;

$$\frac{\partial g_{ab}}{\partial x^c} = g_{ma} \Gamma^m{}_{bc} + g_{mb} \Gamma^m{}_{ac} \tag{21}$$

We may always choose an affine parameterization for a curve $x^\mu(\lambda)$. Then the tangent $v^\mu = dx^\mu / d\lambda$ has constant length,

$$v^\mu v^\nu g_{\mu\nu} = \text{constant} \tag{22}$$

If timelike, then may choose affine parameter τ so that $v^\mu v^\nu g_{\mu\nu} = -1$. Then τ is just proper time along curve.If space like, we can choose proper distance s as parameter, and $v^\mu v^\nu g_{\mu\nu} = +1$.

Recall a spacelike/timelike geodesic is a curve between 2 space-time points with extremal proper length/time. In affine parameterization requiring δs or $\delta \tau = 0$ is equivalent to requiring,

$$L = \int d\lambda \mathcal{L} = \int d\lambda \, g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \tag{23}$$

is extremized. (ie. we can forget the 'square root').

The Euler-Lagrange equations;

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^a}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^a} \tag{24}$$

then give the geodesic condition (in affine parameterization);

$$\frac{d^2 x^j}{d\lambda^2} + \Gamma^j{}_{ik} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda} = 0 \tag{25}$$

Remember that in practice; eg. for FLRW and Schwarzschild it is easier to compute the geodesics by varying this action, rather than using the geodesic equation.

Also recall that you can simplify things by remembering that $\mathcal{L}$ =constant. You can use this to replace one of the E-L equations.

Covariant derivative of a (q,r) tensor $T^{i_1\ldots i_q}{}_{j_1\ldots j_r}$;

$$\begin{aligned}\nabla_a T^{i_1\ldots i_q}{}_{j_1\ldots j_r} \equiv\; & \partial_a T^{i_1\ldots i_q}{}_{j_1\ldots j_r} \\ & +\Gamma^{i_1}{}_{ab} T^{b i_2\ldots i_q}{}_{j_1\ldots j_r} + \Gamma^{i_2}{}_{ab} T^{i_1 b i_3\ldots i_q}{}_{j_1\ldots j_r} + \ldots \\ & -\Gamma^{b}{}_{a j_1} T^{i_1\ldots i_q}{}_{b j_2\ldots j_r} - \Gamma^{b}{}_{a j_2} T^{i_1\ldots i_q}{}_{j_1 b j_3\ldots j_r} + \ldots \end{aligned} \tag{26}$$

This generalises partial derivative - it equals it in LIF - and gives a well defined tensor. Properties;

$$\nabla_a g_{bc} = \nabla_a g^{bc} = 0 \tag{27}$$

Defines notion of parallel transport. We say a vector w^μ is parallel transported along a curve $x(\lambda)$ with tangent v^μ if; $v^\mu \nabla_\mu w^\nu|_{x(\lambda)} = 0$.

Geodesic (in affine parameterization) is curve which transports its tangent along itself;

$$v^i \nabla_i v^j\big|_{x(\lambda)} = 0 \tag{28}$$

Isometries give conservation laws for geodesic motion; Consider inertial particle, proper time τ, tangent $v^\mu = dx^\mu/d\tau$, hence 4-momentum $p^\mu = m v^\mu$. Consider Killing vector k^μ. Then recall one can show (using geodesic eqn and Killing vector equation);

$$\frac{d}{d\tau}\left(p^\mu k_\mu\right) = 0 \tag{29}$$

so $p^\mu k_\mu$ is conserved. If k^μ timelike then $-p^\mu k_\mu$ is the energy of the particle measured by observers following curves with tangent k^μ.

Recall for a function, $[\nabla_\alpha, \nabla_\beta] f = 0$

However, for a convector, $[\nabla_\alpha, \nabla_\beta] v_\mu = R_{\alpha\beta\mu}{}^{\nu} v_\nu$ where $R_{\alpha\beta\nu}{}^{\mu}$ is the *Riemann* $(1,3)$ tensor field, which is;

$$R_{\alpha\beta\mu}{}^{\delta} \;=\; \partial_\beta \Gamma^\delta{}_{\alpha\mu} - \partial_\alpha \Gamma^\delta{}_{\beta\mu} + \Gamma^\nu{}_{\alpha\mu} \Gamma^\delta{}_{\beta\nu} - \Gamma^\nu{}_{\beta\mu} \Gamma^\delta{}_{\alpha\nu} \tag{30}$$

Controls curvature - deviation away from being Mink;

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}\, R_{\mu\alpha\nu\beta}\big|_{x=0}\, x^\alpha x^\beta + O(x^3) \tag{31}$$

Enjoys symmetries;

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta} \tag{32}$$

Ricci tensor, scalar;

$$R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}\,, \quad R = R_\mu{}^\mu \tag{33}$$

and Einstein tensor;

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{34}$$

The Bianchi identity;

$$\nabla_{[\mu} R_{\nu\alpha]\beta}{}^{\rho} = 0 \tag{35}$$

Two contractions give;

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu R^{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0 \tag{36}$$

4.2 Chapters 3, 4 and 5: Special Relativity and matter

Special Relativity is physics in Minkowski space-time.

In Minkowski coordinates; $x^\mu = (t, x, y, z)$, the Minkowski metrc is;

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

The Poincare transformations;

$$x'^{\mu'} = a^{\mu'} + b^{\mu'}{}_{\mu} x^\mu \tag{38}$$

generate all the isometries of Minkowski space-time. Geodesics are straight lines - inertial observers follow geodesics.

Physical laws must be independent of our choice of coordinates; hence they must be given by tensor expressions.

Given a law in Minkowski coordinates, obtain general law by writing variables as tensors and taking $\partial_\mu \to \nabla_\mu$.

Eg. EM;

$$\nabla_\mu F^{\mu\nu} \;=\; j^\nu \quad , \quad \nabla_{[\nu} F_{\alpha\beta]} = 0 \tag{39}$$

where $F_{\mu\nu}$ is an antisymmetric $(0, 2)$ tensor field, and j^μ is a vector field which obeys, $\nabla_\mu j^\mu = 0$.

Newton's laws; for a massive particle with proper time τ following curve tangent $v^\mu = dx^\mu/d\tau$, and hence with 4-momentum $p^\mu = m v^\mu$ for rest mass m;

$$a^\mu = v^\alpha \nabla_\alpha v^\mu \ , \quad f^\mu = m a^\mu \tag{40}$$

In Minkowski coordinates this is equivalent to the more familiar $a^\mu = d^2 x^\mu / d\tau^2$ or alternatively $f^\mu = dp^\mu/d\tau$ (neither of which are tensor expressions, but work only in Minkowski coordinates).

Since any curved metric is Minkowski locally (ie. can go to LIF at some point), on scales smaller than the curvature SR is always recovered in GR.

Any physical law in Minkowski space-time written as a tensor expression simply generalise to a general curved space-time.

Ex. In curved space-time again $a^\mu = v^\alpha \nabla_\alpha v^\mu$ for a particle. A non-accelerating particle therefore follows a geodesic.

Continuous matter is described by a stress tensor; $T_{\mu\nu}$ is a $(0, 2)$ tensor that due to local energy-momentum and angular momentum conservation is **symmetric** $T_{\mu\nu} = T_{\nu\mu}$ and **conserved**; $\nabla^\mu T_{\mu\nu} = 0$.

In the instantaneous rest frame of the matter at some point in space-time (ie. its LIF at that point, chosen so that the total 3-momentum there is zero) then;

- T^{tt} = *rest* mass density of the matter
- $T^{ti} = 0$
- T^{ij} = the usual stress tensor ie. *momentum flux* in direction of x^i through surface normal to x^j.

A perfect fluid is described by local energy density ρ, pressure P and 4-velocity v^μ where $v^\mu v_\mu = -1$ and an 'equation of state' relating P to ρ e.g.;

- Cold matter matter ('Dust') fluid has no pressure, so $P = 0$.
- Hot relativistic matter and radiation has pressure, so $P = \frac{1}{3}\rho$.

Then its stress tensor is,

$$T^{\mu\nu} = (\rho + P)\, u^\mu u^\nu + P\eta^{\mu\nu} \tag{41}$$

Perfect fluids are very simple - the fluid equations of motion are just equivalent to stress-energy conservation, $\nabla^\mu T_{\mu\nu} = 0$.

4.3 Chapters 6 and 8: A geometric origin to gravity and the Einstein equation

Motion in a curved space-time can appear to observers who think space-time is flat as a 'fictitious force' - the 'force of gravity'. This is quite analogous to the 'fictitious centrifugal force' when observers forget that if they are in a rotating frame they are accelerating.

The Einstein equation governs how the matter deforms the space-time. It must be a tensor equation, and must involve the curvature as we wish matter to curve the geometry away from uncurved Minkowski. Stress energy conservation plus the contracted Bianchi then implies the equation should be;

$$G_{\mu\nu} = \kappa \, T_{\mu\nu} \tag{42}$$

Getting the Newtonian limit correct determines the constant κ as;

$$G_{\mu\nu} = 8\pi G_N \, T_{\mu\nu} \tag{43}$$

(in our units $c = 1$).

Now we review the Newtonian limit, or Newtonian space-time. Consider Minkowski space-time, in Minkowski coordinates $x^\mu = (t, x, y, z)$, so $g_{\mu\nu} = \eta_{\mu\nu}$, and deform it a little in a specific way;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \quad \text{with} \quad g_{\mu\nu} = \eta_{\mu\nu} - 2\epsilon \, \Phi \, \delta_{\mu\nu} + O(\epsilon^2) \tag{44}$$

where $|\epsilon| \ll 1$ controls this small deformation. The quantity $\epsilon \, \Phi$ we find to be the usual Newtonian potential.

We calculate quantities to lowest order in ϵ to obtain the Newtonian limit.

Suppose we have a dust fluid in the space-time (so $P = 0$) which is very slowly moving so that $v^t = 1 + O(\epsilon)$ and $v^i = O(\epsilon)$ so we take,

$$T_{\mu\nu} \ = \ \rho \, v_\mu v_\nu \tag{45}$$

and we will require the density is weak so that $\rho \sim O(\epsilon)$.

The Einstein equations can be consistently solved at leading order, and they determine that $\epsilon\Phi$ is simply the Newtonian potential of Newton's gravity, so,

$$\delta^{ij}\partial_i\partial_j\epsilon\Phi = \nabla^2\epsilon\Phi = 4\pi G_N\rho \tag{46}$$

The path of a slow moving timelike geodesic, proper time τ, obeys,

$$\frac{d^2x^i}{d\tau^2} = -\partial_i\epsilon\Phi \tag{47}$$

giving the usual Newton law of gravity.

The proper time of the particle obeys,

$$\frac{dt}{d\tau} = 1 + \epsilon\left(\frac{1}{2}\delta_{ij}v^iv^j - \Phi\right) + \ldots \tag{48}$$

ie. the particle experiences the usual Lorentz time dilation relative to observers at rest (ie. whose proper time is the coordinate time t, so that they must be at constant coordinate location x^i =const), but also a gravitational time dilation effect.

This gives the 'Gravity redshift' effect. A static emitted and observer (at constant $x^i_{(e)}$ and $x^i_{(o)}$) observers have frequencies of their signals emitted and observed related as;

$$\frac{\omega_{(o)}}{\omega_{(e)}} \simeq 1 + \epsilon\frac{\Phi(x_{(e)}) - \Phi(x_{(o)})}{c^2} \tag{49}$$

Also light bending...

4.4 Chapter 10: FLRW and cosmology

For flat spatial geometry, the FLRW metric is;

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right) \tag{50}$$

This is homogeneous and isotropic - all spatial points are the same as any other, and all spatial directions are the same.

Static observers follow curves x^i =const. Their proper time is t.

Solves Einstein equations (which determine scale factor a) if matter is homogeneous and isotropic (ie. stress tensor shares symmetries of metric).

Expansion is measured by rate of change of scale factor; Hubble 'constant' (which is not constant),

$$H \equiv \frac{1}{a}\frac{da}{dt} \tag{51}$$

Also may use conformal time co-ordinate;

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right) = a^2(\tau)\eta_{\mu\nu}dx^\mu dx^\nu \tag{52}$$

Then null geodesics are straight lines in the coordinates τ, x, y, z, at 45^o. Related to time t as,

$$dt = a(\tau)d\tau \ , \quad \text{so} \quad t = \int a(\tau)d\tau \tag{53}$$

Cosmological redshift; A static emitter at time t_e sends a light pulse out, and then another shortly after at time $t_e + \Delta t_e$. These are observed by a static observer at latter times t_o and $t_o + \Delta t_o$ respectively. These are related as;

$$\frac{\Delta t_o}{\Delta t_e} = \frac{a(t_o)}{a(t_e)} \tag{54}$$

Hence in an expanding universe, light observed from far objects appears redshifted (ie. lower frequency than its frequency measured when where is originated from).

4.5 Chapter 11: Schwarzschild

Black hole metric;

$$ds^2 = -\left(1 - \frac{R_S}{r}\right)dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{55}$$

where $d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$ is the line element on the round 2-sphere. We will work with these Schwarschild coordinates $x^\mu = (t, r, \theta, \phi)$

This solves the vacuum ($T_{\mu\nu} = 0$) Einstein equations (for $r > 0$) ie. $R_{\mu\nu} = 0$. It represents the exterior to a spherically symmetric distribution of matter (which in fact must also be static - Birkhoff's theorem). Alternatively the end state of gravitational collapse to a black hole.

$r = R_S$ is the horizon of the black hole; R_S is the 'Schwarzschild radius'. Matter or light entering the horizon may never leave and will be crushed at the singularity at $r = 0$ inevitably a finite time after entering.

Metric looks singular at horizon, but only a 'coordinate singularity'. Using better coordinates (eg. Eddington-Finklestein) the horizon is perfectly regular and locally looks no different to any other place.

In order to deduce, say timelike, geodesics it is best to parameterize using affine parameter proper time τ so,

$$x^\mu(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau)) \tag{56}$$

so the tangent is given by,

$$\frac{dx^\mu(\tau)}{d\tau} = \left(\dot{T}(\tau), \dot{R}(\tau), \dot{\Theta}(\tau), \dot{\Phi}(\tau)\right) \tag{57}$$

where $\dot{T} = dT/d\tau$ and similarly for $\dot{R}, \dot{\Theta}, \dot{\Phi}$.

Then use Euler-Lagrange to find extrema of the Lagrangian;

$$\mathcal{L} \equiv g_{\mu\nu}\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau} = -\left(1 - \frac{R_S}{R}\right)\dot{T}^2 + \left(1 - \frac{R_S}{R}\right)^{-1}\dot{R}^2 + R^2\left(\dot{\Theta}^2 + \sin^2\Theta\dot{\Phi}^2\right)$$

Recall that $\mathcal{L} = -1$; use this rather than one of the Euler-Lagrange equations as it is simpler to work with and equivalent in the end.

Do not try to solve the geodesic equation; $\frac{d^2x^j}{d\lambda^2} + \Gamma^j{}_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda} = 0$. In practice it is always better to compute geodesics in a particular given metric by this variational approach.