STRING THEORY

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ABSTRACT: Part III Course Notes. 24 Lectures.

Contents

1. What is String Theory?

String theory is a theory of strings of a special kind. Consider a string of uniform tension T and uniform mass density ρ, stretched between two points. Small amplitude oscillations of this string propagate along the string with speed

$$v = \sqrt{\frac{T}{\rho}} \, . \tag{1.1}$$

If the string is tightened by pulling on the two ends, the tension goes up and the mass-density decreases slightly, causing v to increase. However, it cannot increase indefinitely because Special Relativity requires $v \le c$, which implies

$$T \le \rho c^2 \, . \tag{1.2}$$

For everyday strings such as violin strings $T \ll \rho c^2$; these are non-relativistic. An example of a string with $T \sim \rho c^2$ is the Schwarzschild black hole viewed as a black string solution of GR in 5 spacetime dimensions; in this case $T = \frac{1}{2}\rho c^2$. The ultra-relativistic case, for which the inequality (1.2) is saturated, i.e. $T = \rho c^2$, is special. These arise as cosmic strings, which are string-like defects in relativistic scalar field theories of relevance to cosmology. Cosmic strings have an internal structure, a core of non-zero size, so they cease to look like strings when probed at wavelengths less than the core size; they are not "elementary" strings.

The strings of String Theory are assumed to be elementary ultra-relativistic strings; they have no internal structure and the only relevant parameter is the string tension T. The oscillation modes of such strings are identified with elementary particles.

1.1 Why study String Theory?

Well, why not? Quantum field theory is, despite the name, essentially a relativistic quantum theory of point particles. Maybe replacing particles by strings will help resolve some of the problems of QFT, for example the non-renormalizability of quantum gravity when considered as a quantum theory of massless spin-2 particles? This turns out to be the case, after inclusion of extra dimensions and supersymmetry.

2. The relativistic point particle

What is an elementary particle?

- (Maths) "A unitary irrep of the Poincaré group". These are classified by mass and spin.

- (Physics) "A particle without structure". The classical action for such a particle should depend only on the geometry of its worldline (plus possible variables describing its spin).

Let's pursue the physicist's answer, in the context of a D-dimensional Minkowski space-time. For zero spin the simplest geometrical action for a particle of mass m is

$$I = -mc^2 \int_A^B d\tau = -mc \int_A^B \sqrt{-ds^2} = -mc \int_{t_A}^{t_B} \sqrt{-\dot{x}^2}\,dt\,, \qquad \dot{x} = \frac{dx}{dt}\,, \qquad (2.1)$$

where t is an arbitrary worldline parameter. In words, the action is the elapsed proper time between an initial point A and a final point B on the particle's worldline.

We could include terms involving the extrinsic curvature K of the worldline, which is essentially the D-acceleration, or yet higher derivative terms, i.e.

$$I = -mc \int dt \sqrt{-\dot{x}^2} \left[1 + \left(\frac{\ell K}{c^2} \right)^2 + \ldots \right]\,, \qquad (2.2)$$

where ℓ is a new length scale, which must be characteristic of some internal structure. In the long-wavelength approximation $c^2 K^{-1} \gg \ell$ this structure is invisible and we can neglect any extrinsic curvature corrections. Or perhaps the particle is truly elementary, and $\ell = 0$. In either case, quantization should yield a Hilbert space carrying a unitary irrep of the Poincaré group. For zero spin this means that the particle's wavefunction Ψ should satisfy the Klein-Gordon equation $(\Box - m^2)\Psi = 0$. There are many ways to see that this is true.

2.1 Gauge invariance

Think of the particle action

$$I[x] = -mc \int dt \sqrt{-\dot{x}^2} \qquad (2.3)$$

as a "1-dim. field theory" for D "scalar fields" $x^m(t)$ ($m = 0, 1, \cdots D - 1$). For a different parameterization, with parameter t', we will have "scalar fields" $x'(t')$, s.t. $x'(t') = x(t)$. If $t' = t - \xi(t)$ for infinitesimal function ξ, then

$$x(t) = x'(t - \xi) = x(t - \xi) + \delta_\xi x(t) = x(t) - \xi \dot{x}(t) + \delta_\xi x(t)\,, \qquad (2.4)$$

and hence

$$\delta_\xi x(t) = \xi(t)\dot{x}(t)\,. \qquad (2.5)$$

This is a gauge transformation with parameter $\xi(t)$. <u>Check</u>:

$$\delta_\xi \sqrt{-\dot{x}^2} = -\frac{1}{\sqrt{-\dot{x}^2}}\, \dot{x} \cdot \frac{d(\delta_\xi x)}{dt} = -\frac{1}{\sqrt{-\dot{x}^2}} \left(\dot{\xi}\dot{x}^2 + \xi \dot{x} \cdot \ddot{x} \right)$$

$$= \dot{\xi}\sqrt{-\dot{x}^2} + \xi \frac{d\sqrt{-\dot{x}^2}}{dt} = \frac{d}{dt}\left(\xi\sqrt{-\dot{x}^2} \right)\,, \qquad (2.6)$$

so the action is invariant for any $\xi(t)$ subject to the b.c.s $\xi(t_A) = \xi(t_B) = 0$. The algebra of these gauge transformations is that of Diff_1, i.e. 1-dim. diffeomorphisms (maths) or 1-dim. general coordinate transformations (phys).

Gauge invariance is **not** a symmetry. Instead it implies a redundancy in the description. We can remove the redundancy by imposing a gauge-fixing condition. For example, in Minkowski space coords. $(x^0, \vec{x})$ we may choose the "temporal gauge"

$$x^0(t) = c\,t\,. \tag{2.7}$$

Since $\delta_\xi x^0 = c\,\xi$ when $x^0 = ct$, insisting on this gauge choice implies $\xi = 0$; i.e. no gauge transformation is compatible with the gauge choice, so the gauge is fixed. In this gauge

$$I = -mc^2 \int dt \sqrt{1 - v^2/c^2} = \int dt \left\{ -mc^2 + \frac{1}{2}mv^2 \left[1 + \mathcal{O}\left(v^2/c^2\right)\right] \right\}, \tag{2.8}$$

where $v = |\dot{\vec{x}}|$. The potential energy is therefore the rest mass energy mc^2, which we can subtract because it is constant. We can then take the $c \to \infty$ limit to get the non-relativistic particle action

$$I_{NR} = \frac{1}{2}m \int dt \, |d\vec{x}/dt|^2\,. \tag{2.9}$$

From now on we set $c = 1$.

2.2 Hamiltonian formulation

If we start from the gauge-invariant action with $L = -m\sqrt{-\dot{x}^2}$, then

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}} \quad \Rightarrow \quad p^2 + m^2 \equiv 0\,. \tag{2.10}$$

So not all components of p are independent, which means that *we cannot solve for $\dot{x}$ in terms of p.* Another problem is that

$$H = \dot{x} \cdot p - L = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} + m\sqrt{-\dot{x}^2} \equiv 0\,, \tag{2.11}$$

so *the canonical Hamiltonian is zero.*

What do we do? Around 1950 Dirac developed methods to deal with such cases. We call the mass-shell condition $p^2 + m^2 = 0$ a "primary" constraint because it is a direct consequence of the definition of conjugate momenta. Sometimes there are "secondary" constraints but we will never encounter them. According to Dirac we should take the Hamiltonian to be the mass-shell constraint times a Lagrange multiplier, so that

$$I = \int dt \left\{ \dot{x} \cdot p - \frac{1}{2}e\left(p^2 + m^2\right) \right\}, \tag{2.12}$$

where $e(t)$ is the Lagrange multiplier. We do not need to develop the ideas that lead to this conclusion because we can easily check the result by eliminating the variables p and e:

- Use the p equation of motion $p = e^{-1}\dot{x}$ to get the new action

$$I[x;e] = \frac{1}{2}\int dt \left\{ e^{-1}\dot{x}^2 - em^2 \right\} . \tag{2.13}$$

  At this point it looks as though we have '1-dim. scalar fields" coupled to 1-dim. "gravity", with "cosmological constant" m^2; in this interpretation e is the square root of the 1-dim. metric, i.e. the "einbein".

- Now eliminate e from (2.13) using the e equation of motion $me = \sqrt{-\dot{x}^2}$, to get the standard point particle action $I = -m \int dt \sqrt{-\dot{x}^2}$.

<u>Elimination lemma</u>. When is it legitimate to solve an equation of motion and substitute the result back into the action to get a new action? Let the action $I[\psi, \phi]$ depend on two sets of variables ψ and ϕ, such that the equation $\delta I / \delta\phi = 0$ can be solved algebraically for the variables ϕ as functions of the variables ψ, i.e. $\phi = \phi(\psi)$. In this case

$$\left. \frac{\delta I}{\delta\phi} \right|_{\phi=\phi(\psi)} \equiv 0 . \tag{2.14}$$

The remaining equations of motion for ψ are then equivalent to those obtained by variation of the new action $\hat{I}[\psi] = I[\psi, \phi(\psi)]$, i.e. that obtained by back-substitution. This follows from the chain rule and (2.14):

$$\frac{\delta\hat{I}}{\delta\psi} = \left. \frac{\delta I}{\delta\psi} \right|_{\phi=\phi(\psi)} + \frac{\delta\phi(\psi)}{\delta\psi} \left. \frac{\delta I}{\delta\phi} \right|_{\phi=\phi(\psi)} = \left. \frac{\delta I}{\delta\psi} \right|_{\phi=\phi(\psi)} . \tag{2.15}$$

Moral: If you use the field equations to eliminate a set of variables then you can substitute the result into the action, to get a new action for the remaining variables, only if the equations you used are those found by varying the original action with respect to the set of variables you eliminate. You can't back-substitute into the action if you use the equations of motion of A to solve for B (although you can still substitute into the remaining equations of motion).

The action (2.12) is still Diff$_1$ invariant. The gauge transformations are now

$$\delta_\xi x = \xi\dot{x} , \qquad \delta_\xi p = \xi\dot{p} , \qquad \delta_\xi e = \frac{d}{dt}(e\xi) . \tag{2.16}$$

However, the action is also invariant under the much simpler gauge transformations

$$\delta_\alpha x = \alpha(t)p , \qquad \delta_\alpha p = 0 , \qquad \delta_\alpha e = \dot{\alpha} . \tag{2.17}$$

Let's call this the "canonical" gauge transformation (for reasons that will become clear). In fact,

$$\delta_\alpha I = \frac{1}{2} \left[\alpha \left(p^2 - m^2 \right) \right]_{t_A}^{t_B} , \qquad (2.18)$$

which is zero if $\alpha(t_A) = \alpha(t_B) = 0$.

The Diff_1 and canonical gauge transformations are equivalent because they differ by a "trivial" gauge transformation.

- Trivial gauge invariances. Consider $I[\psi, \phi]$ again and transformations

$$\delta_f \psi = f \frac{\delta I}{\delta \phi} , \qquad \delta_f \phi = -f \frac{\delta I}{\delta \psi} , \qquad (2.19)$$

  for arbitrary function f. This gives $\delta_f I = 0$, so the action is gauge invariant. As the gauge transformations are zero "on-shell" (i.e. using equations of motion) they have no physical effect. *Any two sets of gauge transformations that differ by a trivial gauge transformation have equivalent physical implications.*

If we fix the gauge invariance by choosing the temporal gauge $x^0(t) = t$ we have

$$\dot{x}^m p_m = \dot{\vec{x}} \cdot \vec{p} - p^0 , \qquad (2.20)$$

so in this gauge the canonical Hamiltonian is

$$H = p^0 = \pm \sqrt{|\vec{p}|^2 + m^2} , \qquad (2.21)$$

where we have used the constraint to solve for p^0. The sign ambiguity is typical for a relativistic particle.

The canonical Hamiltonian *depends on the choice of gauge*. Another possible gauge choice is *light-cone gauge*. Choose phase-space coordinates

$$x^\pm = \frac{1}{\sqrt{2}} \left(x^1 \pm x^0 \right) , \qquad \mathbf{x} = \left(x^2, \dots, x^{D-1} \right)$$

$$p_\pm = \frac{1}{\sqrt{2}} \left(p_1 \pm p_0 \right) , \qquad \mathbf{p} = \left(p_2, \dots, p_{D-1} \right) . \qquad (2.22)$$

Then

$$\dot{x}^m p_m = \dot{x}^+ p_+ + \dot{x}^- p_- + \dot{\mathbf{x}} \cdot \mathbf{p} , \qquad p^2 \equiv \eta^{mn} p_m p_n = 2 p_+ p_- + |\mathbf{p}|^2 . \qquad (2.23)$$

The latter equation follows from the fact that the non-zero components of the Minkowski metric in light-cone coordinates are

$$\eta^{+-} = \eta^{-+} = 1 = \eta_{-+} = \eta_{+-} , \qquad \eta^{IJ} = \eta_{IJ} = \delta_{IJ} , \quad I, J = 1, \dots D - 2. \qquad (2.24)$$

It also follows from this that

$$p^+ = p_- , \qquad p^- = p_+ . \qquad (2.25)$$

The light-cone gauge is

$$x^+(t) = t.$$

(2.26)

Since $\delta_\alpha x^+ = \alpha p^+ = \alpha p_-$ the gauge is fixed provided that $p_- \neq 0$. In this gauge

$$\dot{x}^m p_m = \dot{\mathbf{x}} \cdot \mathbf{p} + \dot{x}^- p_- + p_+,$$

(2.27)

so the canonical Hamiltonian is now

$$H = -p_+ = \frac{|\mathbf{p}|^2 + m^2}{2p_-},$$

(2.28)

where we have used the mass-shell constraint to solve for p_+.

- *Poisson brackets.* For mechanical model with action

$$I[q, p] = \int dt \left[\dot{q}^I p_I - H(q, p) \right]$$

(2.29)

the Poisson bracket of any two functions (f, g) on phase space is

$$\{f, g\}_{PB} = \frac{\partial f}{\partial q^I} \frac{\partial g}{\partial p_I} - \frac{\partial f}{\partial p_I} \frac{\partial g}{\partial q^I}.$$

(2.30)

In particular,

$$\left\{ q^I, p_J \right\}_{PB} = \delta^I_J.$$

(2.31)

- *More generally*, we start from a symplectic manifold, a phase-space with coordinates z^A and a symplectic (closed, invertible) 2-form $\Omega = \frac{1}{2} \Omega_{AB} \, dz^A \wedge dz^B$. Locally, since $d\Omega = 0$,

$$\Omega = d\omega, \qquad \omega = dz^A f_A(z),$$

(2.32)

and the action in local coordinates is

$$I = \int dt \left[\dot{z}^A f_A(z) - H(z) \right].$$

(2.33)

The PB of functions (f, g) is defined as

$$\{f, g\}_{PB} = \Omega^{AB} \frac{\partial f}{\partial z^A} \frac{\partial g}{\partial z^B},$$

(2.34)

where Ω^{AB} is the inverse of Ω_{AB}. The PB is an antisymmetric bilinear product, from its definition. Also, for any three functions (f, g, h),

$$d\Omega = 0 \quad \Leftrightarrow \quad \{\{f, g\}_{PB}, h\}_{PB} + \quad \text{cyclic permutations} \equiv 0.$$

(2.35)

In other words, the PB satisfies the Jacobi identity, and is therefore a Lie bracket, as a consequence of the closure of the symplectic 2-form.

- *Darboux theorem.* This states that there exist local coordinates such that

$$\Omega = dp_I \wedge dq^I \quad \Rightarrow \quad \omega = p_I dq^I + d() \, . \tag{2.36}$$

This leads to the definition (2.30) of the PB.

- *Canonical transformations.* Any function Q on phase-space is the generator of an infinitesimal change of phase-space coordinates; its action on any function f of the coordinates is

$$\delta_\epsilon f = \{f, Q\}_{PB} \, \epsilon \tag{2.37}$$

Suppose that we have Darboux coordinates (q^I, p_I). Then

$$\delta_\epsilon q^I = \epsilon \frac{\partial Q}{\partial p_I} \qquad \delta_\epsilon p_I = -\epsilon \frac{\partial Q}{\partial q^I} \, . \tag{2.38}$$

Notice that

$$\delta_\epsilon \left(dp_I \wedge dq^I \right) = \epsilon \left[dp_I \wedge d\left(\frac{\partial Q}{\partial p_I} \right) + d\left(\frac{\partial Q}{\partial q^I} \right) \wedge dq^I \right] = 0 \, . \tag{2.39}$$

The last equality follows from the symmetry of mixed partial derivatives. In other words, the transformation generated by Q preserves the form of the symplectic 2-form, equivalently the Poisson bracket. Such transformations are called *symplectic diffeomorphisms* (Maths) or *canonical transformations* (Phys.)

2.2.1 Gauge invariance and first-class constraints

Consider the action

$$I = \int dt \left\{ \dot{q}^I p_I - \lambda^i \varphi_i(q, p) \right\} \, , \qquad I = 1, \ldots, N \quad ; \quad i = 1, \ldots, n < N \, . \tag{2.40}$$

The Lagrange multipliers λ^i impose the phase-space constraints $\varphi_i = 0$. Let us suppose that

$$\{\varphi_i, \varphi_j\}_{PB} = f_{ij}{}^k \varphi_k \tag{2.41}$$

for some phase-space *structure functions* $f_{ij}{}^k = -f_{ji}{}^k$. In this case we say that the constraints are "first-class" (Dirac's terminology. There may be "second-class" constraints, but we don't need to know about that now). The special feature of first-class constraints is that *they generate gauge invariances.*

<u>Lemma</u>. The canonical transformation generated by a function $Q(q, p)$ is such that

$$\delta_\epsilon \left(\dot{x}^I p_I \right) = \dot{\epsilon} Q + \frac{d}{dt} \left[\epsilon \left(p_I \frac{\partial Q}{\partial p_I} - Q \right) \right] \tag{2.42}$$

when the infinitesimal parameter ϵ is allowed to be an arbitrary function of the worldline parameter t. Proof: Exercise [N.B. for constant ϵ the variation must be a total derivative because the transformation is canonical.]

Applying this for $\epsilon Q = \epsilon^i \varphi_i$ we get

$$\delta_\epsilon \left(\dot{q}^I p_I \right) = \dot{\epsilon}^i \varphi_i + \frac{d}{dt} \, () \, , \tag{2.43}$$

and we also have

$$\delta_\epsilon \left(\lambda^i \varphi_i \right) = \delta_\epsilon \lambda^i \varphi_i + \lambda^i \epsilon^j \left\{ \varphi_i, \varphi_j \right\}_{PB} = \left(\delta_\epsilon \lambda^k + \lambda^i \epsilon^j f_{ij}{}^k \right) \varphi_k \, , \tag{2.44}$$

where we use (2.41) in the second equality. Putting these result together, we have

$$\delta_\epsilon I = \int dt \left\{ \left(\dot{\epsilon}^k - \delta_\epsilon \lambda^k - \lambda^i \epsilon^j f_{ij}{}^k \right) \varphi_k + \frac{d}{dt} \, () \right\} . \tag{2.45}$$

As the Lagrange multipliers are not functions of canonical variables, their transformations can be chosen independently. If we choose

$$\delta_\epsilon \lambda^k = \dot{\epsilon}^k + \epsilon^i \lambda^j f_{ij}{}^k \, , \tag{2.46}$$

then $\delta_\epsilon I$ is a surface term, which is zero if we impose the b.c.s $\epsilon^i(t_A) = \epsilon^i(t_B) = 0$.

The point particle is a very simple (abelian) example. The one constraint is

$$\varphi = \frac{1}{2} \left(p^2 + m^2 \right) \, , \tag{2.47}$$

and it is trivially first-class. It generates the canonical gauge transformations:

$$\delta_\alpha x = \frac{1}{2} \alpha \left\{ x, p^2 + m^2 \right\}_{PB} = \alpha p \, ,$$

$$\delta_\alpha p = \frac{1}{2} \alpha \left\{ p, p^2 + m^2 \right\}_{PB} = 0 \, , \tag{2.48}$$

and if we apply the formula (2.46) to get the gauge transformation of the einbein, we find that $\delta_\alpha e = \dot{\alpha}$.

The general model (2.40) also includes the string, as we shall see later. This is still a rather simple case because the structure functions are constants, which means that the constraint functions φ_i span a (non-abelian) Lie algebra. In such cases the transformation (2.46) is a Yang-Mills gauge transformation for a 1-dim. YM gauge potential.

2.2.2 Gauge fixing

We can fix the gauge generated by a set of n first-class constraints by imposing n gauge-fixing conditions

$$\chi^i(q, p) = 0 \qquad i = 1, \dots, n \, . \tag{2.49}$$

The gauge transformation of these constraints is

$$\delta_\epsilon \chi^i = \left\{ \chi^i, \varphi_j \right\}_{PB} \epsilon^j \, , \tag{2.50}$$

so if we want $\delta_\epsilon \chi^i = 0$ to imply $\epsilon^j = 0$ for all j (which is exactly what we do want in order to fix the gauge completely) then we must choose the functions χ^i such that

$$\det \left\{ \chi^i, \varphi_j \right\}_{PB} \neq 0 \, . \tag{2.51}$$

This is a useful test for any proposed gauge fixing condition.

In addition to requiring that the gauge-fixing conditions $\chi^i = 0$ actually do fix the gauge, it should also be possible to make a gauge transformation to ensure that $\chi^i = 0$ if this is not already the case. In particular, if $\chi^i = f^i$ for infinitesimal functions f^i , and $\hat{\chi}^i = \chi^i + \delta_\epsilon \chi^i$, then $\hat{\chi}^i = f^i + \delta_\epsilon \chi^i$ and we should be able to find parameters ϵ^i such that $\hat{\chi}^i = 0$. This requires us to solve the equation

$$\left\{ \chi^i, \varphi_j \right\}_{PB} \epsilon^j = -f^i \tag{2.52}$$

for ϵ^i, but a solution exists for arbitrary f^i iff the matrix $\{\chi^i, \varphi_j\}_{PB}$ has non-zero determinant.

<u>Corollary</u>. Whenever $\{\chi^i, \varphi_j\}_{PB}$ has zero determinant, two problems arise. One is that the gauge fixing conditions don't completely fix the gauge, and the other is that you can't always arrange for the gauge fixing conditions to be satisfied by making a gauge transformation. This is a *very general point*. Consider the Lorenz gauge $\partial \cdot A = 0$ in electrodynamics (yes, that's Ludwig Lorenz, not Henrik Lorentz of the Lorentz transformation). A gauge transformation $A \rightarrow A + d\alpha$ of the gauge condition gives $\Box \alpha = 0$, which does *not* imply that $\alpha = 0$; the gauge has not been fixed completely. It is also true, and for the same reason, that you can't always make a gauge transformation to get to the Lorenz gauge if $\partial \cdot A$ is not zero, even if it is arbitrarily close to zero: the reason is that the operator $\Box$ is not invertible because there are non-zero solutions of the wave equation that cannot be eliminated by imposing the b.c.s permissible for hyperbolic partial differential operators. The Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ does not have this problem because ∇^2 is invertible for appropriate b.c.s (but it breaks manifest Lorentz invariance).

The same problem will arise if we try to fix the gauge invariance of the action (2.40) by imposing conditions on the Lagrange multipliers. More on this later.

2.2.3 Continuous symmetries and Noether's theorem

In addition to its gauge invariance, the point particle action is invariant (in a Minkowski background) under the Poincaré transformations

$$\delta_\Lambda X^m = A^m + \Lambda^m{}_n X^n \, , \qquad \delta_\Lambda P_m = \Lambda_m{}^n P_n \, , \tag{2.53}$$

where A^m and $\Lambda_{mn} = -\Lambda_{nm}$ are constant parameters of a spacetime translation and Lorentz transformation, respectively. Noether's theorem implies that there are associated constants of the motion, i.e. conserved charges. These can be found easily using the following version of Noether's theorem:

- **Noether's Theorem**. Let $I[\phi]$ be an action functional invariant under an infinitesimal transformation $\delta_\epsilon \phi$ for constant parameter ϵ. Then its variation when ϵ is an *arbitrary* function of t must be of the form

$$\delta_\epsilon I = \int dt\, \dot\epsilon\, Q\,. \tag{2.54}$$

The quantity Q is a constant of motion. To see this, choose $\epsilon(t)$ to be zero at the endpoints of integration. In this case, integration by parts gives us

$$\delta_\epsilon I = -\int dt\, \epsilon\, \dot Q\,. \tag{2.55}$$

But the left hand side is zero if we use the field equations because these extremize the action for *any* variation of ϕ, whereas the right-hand side is zero for any $\epsilon(t)$ (with the specified endpoint conditions) only if $\dot Q(t) = 0$ for any time t (within the integration limits).

This proves Noether's theorem: a continuous symmetry implies a conserved charge (i.e. constant of the motion); it has to be continuous for us to be able to consider its infinitesimal form. The proof is constructive in that it also gives us the corresponding Noether charge: it is Q. Also, given Q we can recover the symmetry transformation from the formula $\delta_\epsilon \phi = \{\phi, \epsilon Q\}_{PB}$. There may be conserved charges for which the RHS of this formula is zero. These are "topological charges", which do not generate symmetries; they are not Noether charges.

To apply this proof of Noether's theorem to Poincaré invariance of the point particle action, we allow the parameters A and $\Lambda^m{}_n$ of (2.53) to be time-dependent. A calculation then shows that

$$\delta I = \int dt \left\{ \dot A^m \mathcal{P}_m + \frac{1}{2}\dot\Lambda^n{}_m \mathcal{J}^m{}_n \right\}\,, \tag{2.56}$$

where

$$\mathcal{P}_m = P_m\,, \qquad \mathcal{J}^m{}_n = X^m P_n - X_n P^m\,, \tag{2.57}$$

which are therefore the Poincaré charges. Notice that they are gauge-invariant; this is obvious for $\mathcal{P}_m$, and for $\mathcal{J}^m{}_n$ we have

$$\delta_\alpha \mathcal{J}^m{}_n = \alpha \left(P^m P_n - P_n P^m \right) = 0\,. \tag{2.58}$$

Gauge-fixing and symmetries. If we have fixed a gauge invariance by imposing gauge-fixing conditions $\chi^i = 0$, then what happens if our gauge choice does not respect a symmetry with Noether charge Q, i.e. what happens if $\{Q, \chi^i\}_{PB}$ is non-zero.

The answer is that the symmetry is *not* broken. The reason is that there is an intrinsic ambiguity in the symmetry transformation generated by Q whenever there are gauge invariances. We may take the symmetry transformation to be

$$\delta_\epsilon f = \{f, Q\}_{PB}\, \epsilon + \{f, \varphi_j\}_{PB}\, \alpha^j(\epsilon)\,. \tag{2.59}$$

That is, a symmetry transformation with parameter ϵ combined with a gauge transformtion for which the parameters α^i are fixed, in a way to be determined, in terms of ϵ. Because gauge transformations have no physical effect, such a transformation is as good as the one generated by Q alone. The parameters $\alpha^i(\epsilon)$ are determined by requiring that the modified symmetry transformation respect the gauge conditions $\chi^i = 0$, i.e.

$$0 = \{\chi^i, Q\}_{PB}\, \epsilon + \{\chi^i, \varphi_j\}_{PB}\, \alpha^j(\epsilon)\,. \tag{2.60}$$

As long as $\{\chi^i, \varphi_j\}_{PB}$ has non-zero determinant, we can solve this equation for all α^i in terms of ϵ.

Moral: gauge-fixing never breaks symmetries, because it just removes redundancies. If a symmetry *is* broken by some gauge choice then there is something wrong with the gauge choice!

2.2.4 Quantization: canonical and Dirac's method

We will use the prescription

$$\left\{q^I, p_J\right\}_{PB} \to -\frac{i}{\hbar}\left[\hat{q}^I, \hat{p}_J\right]\,, \tag{2.61}$$

which gives us the canonical commutation relations for the operators $\hat{q}^I$ and $\hat{p}_I$ that replace the classical phase-space coordinates:

$$\left[\hat{q}^I, \hat{p}_J\right] = i\hbar\delta^I_J\,. \tag{2.62}$$

Let's apply this to the point particle in temporal gauge. In this case the canonical commutation relations are precisely (2.62) where $I = 1, \ldots, D-1$. We can realise this on eigenfunctions of $\hat{x}^I$, with e-value x^I, by setting $\hat{p}_I = -i\hbar\partial_I$. The Schroedinger equation is

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t}\,, \qquad H = \pm\sqrt{-\hbar^2\nabla^2 + m^2}\,. \tag{2.63}$$

Iterating we deduce that

$$\left[-\nabla^2 + \partial_t^2 + (m/\hbar)^2\right]\Psi(x) = 0\,. \tag{2.64}$$

Since $t = x^0$, this is the Klein-Gordon equation for a scalar field Ψ and mass parameter $m/\hbar$ (the mass parameter of the field equation is the particle mass divided by $\hbar$). The final result is Lorentz invariant even though this was not evident at each step.

An alternative procedure is provided by Dirac's method for quantization of systems with first-class constraints. We'll use the point particle to illustrate the idea.

- Step 1. We start from the manifestly Lorentz invariant, but also gauge invariant, action, and we quantise as if there were no constraint. This means that we have the canonical commutation relations

$$[\hat{q}^m, \hat{p}_n] = i\hbar \delta^m_n \,. \qquad (2.65)$$

We can realise this on eigenfunctions $\Psi(x)$ of $\hat{x}^m$ by setting $\hat{p}_m = -i\hbar\partial_m$.

- Step 2. Because of the gauge invariance there are unphysical states in the Hilbert space. We need to remove these with a constraint. The mass-shell constraint encodes the full dynamics of the particle, so we now impose this in the quantum theory as the *physical state condition*

$$\left(\hat{p}^2 + m^2\right)|\Psi\rangle = 0 \,. \qquad (2.66)$$

This is equivalent to the Klein-Gordon equation

$$\left[\Box_D - (m/\hbar)^2\right]\Psi(x) = 0 \,, \qquad \Psi(x) = \langle x|\Psi\rangle \,. \qquad (2.67)$$

where $\Box_D = \eta^{mn}\partial_m\partial_n$ is the wave operator in D-dimensions.

More generally, for the general model with first-class constraints, we impose the physical state conditions

$$\hat{\varphi}_i|\Psi\rangle = 0 \,, \quad i = 1, \ldots, n \,. \qquad (2.68)$$

The consistency of these conditions requires that

$$[\hat{\varphi}_i, \hat{\varphi}_j]|\Psi\rangle = 0 \quad \forall\, i, j \,. \qquad (2.69)$$

This would be guaranteed if we could apply the PB-to-commutator prescription of (2.61) to arbitrary phase-space functions, because this would give

$$[\hat{\varphi}_i, \hat{\varphi}_j] = i\hbar f_{ij}{}^k \hat{\varphi}_k \,, \qquad (?) \qquad (2.70)$$

and the RHS annihilates physical states. However, because of operator ordering ambiguities there is no guarantee that (2.70) will be true when the functions φ_i are non-linear. We can use some of the ambiguity to redefine what we mean by $\hat{\varphi}_i$, but this may not be sufficient. *There could be a quantum anomaly.* The string will provide an example of this.

From now on we set $\hbar = 1$.

3. The Nambu-Goto string

The string analog of a particle's worldline is its "worldsheet": the 2-dimensional surface in spacetime that the string sweeps out in the course of its time evolution. Strings can be *open*, with two ends, or *closed*, with no ends. We shall start by considering a closed string. This means that the parameter σ specifying position on the string is subject to a periodic identification. The choice of period has no physical significance; we will choose it to be 2π; i.e. ($\sim$ means "is identified with")

$$\sigma \sim \sigma + 2\pi\,. \tag{3.1}$$

The worldsheet of a closed string is topologically a cylinder, parametrised by σ and some arbitrary time parameter t. We can consider these together as σ^μ ($\mu = 0, 1$), i.e.

$$\sigma^\mu = (t, \sigma). \tag{3.2}$$

The map from the worldsheet to Minkowski space-time is specified by worldsheet fields $X^m(t, \sigma)$. Using this map we can pull back the Minkowski metric on space-time to the worldsheet to get the induced worldsheet metric

$$g_{\mu\nu} = \partial_\mu X^m \partial_\nu X^n \eta_{mn}\,. \tag{3.3}$$

The natural string analog of the point particle action proportional to the proper length of the worldline (i.e. the elapsed proper time) is the Nambu-Goto action, which is proportional to the area of the worldsheet in the induced metric, i.e.

$$I_{NG} = -T \int dt \oint d\sigma \sqrt{-\det g}\,, \tag{3.4}$$

where the constant T is the string tension. Varying with respect to X we get the NG equation of motion

$$\partial_\mu \left(\sqrt{-\det g}\, g^{\mu\nu} \partial_\nu X \right) = 0\,. \tag{3.5}$$

This is just the 2-dimensional massless wave equation for a set of scalar fields $\{X^m\}$ (scalars with respect to the 2D local Lorentz group) propagating on a 2-dimensional spacetime, but with a metric g that depends on the scalar fields.

Denoting derivatives with respect to t by an overdot and derivatives with respect to σ by a prime, we have

$$g_{\mu\nu} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix}\,, \tag{3.6}$$

and hence the following alternative form of the NG action

$$I_{NG} = -T \int dt \oint d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}\,, \tag{3.7}$$

where
$$X(t, \sigma + 2\pi) = X(t, \sigma).$$ (3.8)

This action is Diff$_2$ invariant; i.e. invariant under arbitrary local reparametrization of the worldsheet coordinates. From an active point of view (transform fields rather than the coordinates) a Diff$_2$ transformation of X is

$$\delta_\zeta X^m = \zeta^\mu \partial_\mu X^m,$$ (3.9)

where $\zeta(t, \sigma)$ is an infinitesimal worldsheet vector field. This implies that

$$\delta_\zeta \left(\sqrt{-\det g} \right) = \partial_\mu \left(\zeta^\mu \sqrt{-\det g} \right),$$ (3.10)

and hence that the action is invariant if ζ is zero at the initial and final times.

3.1 Hamiltonian formulation

The worldsheet momentum density $P_m(t, \sigma)$ canonically conjugate to the wordsheet fields $X^m(t, \sigma)$ is

$$P_m = \frac{\delta L}{\delta \dot{X}^m}, \qquad L = -T \oint d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2},$$ (3.11)

which gives

$$P_m = \frac{T}{\sqrt{-\det g}} \left[\dot{X}_m X'^2 - X'_m \left(\dot{X} \cdot X' \right) \right].$$ (3.12)

This implies the following identities

$$P^2 + (TX')^2 \equiv 0, \qquad X'^m P_m \equiv 0.$$ (3.13)

In addition, the canonical Hamiltonian is

$$H = \oint d\sigma \dot{X}^m P_m - L \equiv 0.$$ (3.14)

As for the particle, we should take the Hamiltonian to be a sum of Lagrange multipliers times the constraints, so we should expect the phase-space form of the action to be

$$I = \int dt \oint d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left[P^2 + (TX')^2 \right] - u X'^m P_m \right\},$$ (3.15)

where $e(t, \sigma)$ and $u(t, \sigma)$ are Lagrange multipliers (analogous to the "lapse" and "shift" functions appearing in the Hamiltonian formulation of GR). To check this, we eliminate P by using its equation of motion:

$$P = e^{-1} D_t X, \qquad D_t X \equiv \dot{X} - uX'.$$ (3.16)

We are assuming here that e is nowhere zero (but we pass over this point). Back substitution takes us to the action

$$I = \frac{1}{2} \int dt \oint d\sigma \left\{ e^{-1} (D_t X)^2 - e (TX')^2 \right\} . \tag{3.17}$$

Varying u in this new action we find that

$$u = \frac{\dot{X} \cdot X'}{X'^2} \quad \Rightarrow \quad D_t X^2 = \frac{\det g}{X'^2} . \tag{3.18}$$

Here we assume that X'^2 is non-zero (but we pass over this point too). Eliminating u we arrive at the action

$$I = \frac{1}{2} \int dt \oint d\sigma \left\{ e^{-1} \frac{\det g}{X'^2} - e (TX')^2 \right\} . \tag{3.19}$$

Varying this action with respect to e we find that

$$Te = \sqrt{-\det g}/X'^2 , \tag{3.20}$$

and back-substitution returns us to the Nambu-Goto action in its original form.

3.1.1 Alternative form of phase-space action

Notice that the phase-space constraints are equivalent to

$$\mathcal{H}_\pm = 0 , \qquad \mathcal{H}_\pm \equiv \frac{1}{4T} (P \pm TX')^2 , \tag{3.21}$$

so we may rewrite the action as

$$I = \int dt \oint d\sigma \left\{ \dot{X}^m P_m - \lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+ \right\} , \qquad \lambda^\pm = Te \pm u . \tag{3.22}$$

3.1.2 Gauge invariances

From the Hamiltonian form of the NG string action (3.15) we read off the canonical Poisson bracket relations[1]

$$\{X^m(\sigma), P_n(\sigma')\}_{PB} = \delta_n^m \, \delta(\sigma - \sigma') . \tag{3.23}$$

Using this one may now compute the PBs of the constraint functions. One finds that

$$\begin{aligned}
\{\mathcal{H}_+(\sigma), \mathcal{H}_+(\sigma')\}_{PB} &= [\mathcal{H}_+(\sigma) + \mathcal{H}_+(\sigma')] \, \delta'(\sigma - \sigma') , \\
\{\mathcal{H}_-(\sigma), \mathcal{H}_-(\sigma')\}_{PB} &= -[\mathcal{H}_-(\sigma) + \mathcal{H}_-(\sigma')] \, \delta'(\sigma - \sigma') , \\
\{\mathcal{H}_+(\sigma), \mathcal{H}_-(\sigma')\}_{PB} &= 0 .
\end{aligned} \tag{3.24}$$

This shows that

---

[1]We can put the action into the form (2.40) by expressing the worldsheet fields as Fourier series; we will do this later. Then we can read off the PBs of the Fourier components, and use them to get the PBs of the worldsheet fields. The result is as given.

- The constraints are "first-class", with constant structure functions, which are therefore the structure constants of a Lie algebra

- This Lie algebra is a direct sum of two isomorphic algebras ($-\mathcal{H}_-$ obeys the same algebra as $\mathcal{H}_+$). In fact, it is the algebra

$$\text{Diff}_1 \oplus \text{Diff}_1 . \qquad (3.25)$$

We will verify this later. Notice that this is a proper subalgebra of Diff_2. Only the $\text{Diff}_1 \oplus \text{Diff}_1$ subalgebra has physical significance because all other gauge transformations of Diff_2 are "trivial" in the sense explained earlier for the particle.

The gauge transformation of any function F on phase space is

$$\delta_\xi F = \left\{ F, \oint d\sigma \left(\xi^- \mathcal{H}_- + \xi^+ \mathcal{H}_+ \right) \right\}_{PB} . \qquad (3.26)$$

where $\xi^\pm$ are arbitrary parameters. This gives

$$\delta X = \frac{1}{2T} \xi^- \left(P - TX' \right) + \frac{1}{2T} \xi^+ \left(P + TX' \right) ,$$
$$\delta P = -\frac{1}{2} \left[\xi^- \left(P - TX' \right) \right]' + \frac{1}{2} \left[\xi^+ \left(P + TX' \right) \right]' . \qquad (3.27)$$

Notice that

$$\delta_{\xi^-} \left(P + TX' \right) = 0 , \qquad \delta_{\xi^+} \left(P - TX' \right) = 0 , \qquad (3.28)$$

and hence $\delta_{\xi^\mp} \mathcal{H}_\pm = 0$, as expected from the fact that the algebra is a direct sum ($\mathcal{H}_+$ has zero PB with $\mathcal{H}_-$).

To get invariance of the action we have to transform the Lagrange multipliers too. One finds that

$$\delta \lambda^- = \dot{\xi}^- + \lambda^- \left(\xi^- \right)' - \xi^- \left(\lambda^- \right)' , \qquad \delta \lambda^+ = \dot{\xi}^+ - \lambda^+ \left(\xi^+ \right)' + \xi^+ \left(\lambda^+ \right)' . \qquad (3.29)$$

We see that $\lambda^\pm$ is a gauge potential for the $\xi^\pm$-transformation, with each being inert under the gauge transformation associated with the other, as expected from the direct sum structure of the gauge algebra. Notice the sign differences in these two transformations; they are a consequence of the fact that $\mathcal{H}_+$ has the same PB algebra as $-\mathcal{H}_-$.

3.1.3 Symmetries of NG action

The closed NG action has manifest Poincaré invariance, with Noether charges

$$\mathcal{P}_m = \oint d\sigma P_m , \qquad \mathcal{J}_{mn} = 2 \oint d\sigma X_{[m} P_{n]} . \qquad (3.30)$$

These are constants of the motion. [Exercise: verify that the NG equations of motion imply that $\dot{\mathcal{P}}_m = 0$ and $\dot{\mathcal{J}}_{mn} = 0$.]

N.B. We use the following notation

$$T_{[mn]} = \frac{1}{2}\left(T_{mn} - T_{nm}\right) , \qquad T_{(mn)} = \frac{1}{2}\left(T_{mn} + T_{nm}\right) . \tag{3.31}$$

In other words, we use square brackets for antisymmetrisation and round brackets for symmetrisation, in both cases with "unit strength" (which means, for tensors of any rank, that $A_{[m_1 \ldots m_n]} = A_{m_1 \ldots m_n}$ if A is totally antisymmetric, and $S_{(m_1 \ldots m_n)} = S_{m_1 \ldots m_n}$ if S is totally symmetric).

The closed NG string is also invariant under **worldsheet parity**: $\sigma \to -\sigma$ (mod 2π). The worldsheet fields (X, P) are parity even, which means that X' is parity odd and hence $(P + TX')$ and $(P - TX')$ are exchanged by parity. This implies that $\mathcal{H}_\pm$ are exchanged by parity.

3.2 Monge gauge

A natural analogue of the temporal gauge for the particle is a gauge in which we set not only $X^0 = t$, to fix the time-reparametrization invariance, but also (say) $X^1 = \sigma$, to fix the reparametrization invariance of the string[2]. This is often called the "static gauge" but this is not a good name because there is no restriction to static configurations. A better name is "Monge gauge", after the 18th century French geometer who used it in the study of surfaces. So, the Monge gauge for the NG string is

$$X^0(t, \sigma) = t \qquad X^1(t, \sigma) = \sigma . \tag{3.32}$$

In this gauge the action (3.15) becomes

$$I = \int dt \oint d\sigma \left\{ \dot{X}^I P_I + P_0 - u\left(P_1 + X'^I P_I\right) \right.$$
$$\left. - \frac{1}{2}e\left[-P_0^2 + P_1^2 + |\mathbf{P}|^2 + T^2\left(1 + |\mathbf{X}'|^2\right)\right] \right\} , \tag{3.33}$$

where $I = 2, \ldots, D - 2$, and $\mathbf{X}$ is the $(D - 2)$-vector with components X^I (and similarly for $\mathbf{P}$). We may solve the constraints for P_1, and P_0^2. Choosing the sign of P_0 corresponding to positive energy, we arrive at the action

$$I = \int dt \oint d\sigma \left\{ \dot{X}^I P_I - T\sqrt{1 + |\mathbf{X}'|^2 + T^{-2}\left[|\mathbf{P}|^2 + (\mathbf{X}' \cdot \mathbf{P})^2\right]} \right\} . \tag{3.34}$$

---

[2]We could choose any linear combination of the space components of X to equal σ but locally we can always orient the axes such that this combination equals X^1.

The expression for the Hamiltonian in Monge gauge simplifies if $\mathbf{P}$ is momentarily zero; we then have

$$H = T \oint d\sigma \sqrt{1 + |\mathbf{X}'|^2} \qquad (\mathbf{P} = \mathbf{0}). \tag{3.35}$$

The integral equals the proper length L of the string. To see this, we observe that the induced worldsheet metric in Monge gauge is

$$\begin{aligned}
ds^2\big|_{\text{ind}} &= -dt^2 + d\sigma^2 + \left|\dot{\mathbf{X}}dt + \mathbf{X}'d\sigma\right|^2 \\
&= -\left(1 - |\dot{\mathbf{X}}|^2\right)dt^2 + 2\dot{\mathbf{X}} \cdot \mathbf{X}'d\sigma dt + \left(1 + |\mathbf{X}'|^2\right)d\sigma^2,
\end{aligned} \tag{3.36}$$

and hence

$$L = \oint d\sigma \sqrt{ds^2|_{\text{ind}}(t = \text{const.})} = \oint d\sigma \sqrt{1 + |\mathbf{X}'|^2}. \tag{3.37}$$

Also, when $\mathbf{P} = \mathbf{0}$ the equations of motion in Monge gauge imply that $\dot{\mathbf{X}} = \mathbf{0}$, so the string is momentarily at rest. The energy of such a string is $H = TL$, and hence the (potential) energy per unit length, or energy density, of the string is

$$\mathcal{E} = T, \tag{3.38}$$

as expected for an ultra-relativistic string.

An ultra-relativistic string cannot support tangential momentum. Given $X^0 = t$, the constraint $X' \cdot P = 0$ becomes

$$\vec{X}' \cdot \vec{P} = 0, \tag{3.39}$$

which tells us that the (space) momentum density at any point on the string is orthogonal to the tangent to the string at that point; the momentum density has no tangential component. This has various consequences. One is that there can be no longitudinal waves on the string (i.e. sound waves). Only transverse fluctuations are physical.

Another consequence is that a plane circular loop of NG string cannot be supported against collapse by rotation in the plane (which can be done if $T < \mathcal{E}$). This does not mean that a plane circular loop of string cannot be supported against collapse by rotation in other planes; we'll see an example later.

3.3 Conformal gauge

We may also fix the gauge by imposing conditions on the Lagrange multipliers $\lambda^\pm$, which are our gauge fields. The *conformal gauge* is

$$\lambda^+ = \lambda^- = 1. \tag{3.40}$$

In the course of proving equivalence of the phase-space action to the NG action, we found expressions for Te and u, which are equivalent to

$$\lambda^{\pm} = Te \pm u = \frac{\sqrt{-\det g} \pm \dot{X} \cdot X'}{(X')^2} \, . \tag{3.41}$$

Setting $\lambda^{\pm} = 1$ we deduce that

$$(X')^2 \mp \dot{X} \cdot X' = \sqrt{-\det g} \, . \tag{3.42}$$

Squaring, and then simplifying the result yields

$$(X')^2 \left(\dot{X} \mp X' \right)^2 = 0 \, . \tag{3.43}$$

As we are supposing that $(X')^2 \neq 0$, we deduce that

$$\dot{X}^2 + (X')^2 = 0 \quad \& \quad \dot{X} \cdot X' = 0 \tag{3.44}$$

and hence, from (3.6),

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \qquad \left[\Omega^2 = (X')^2 \right] \, . \tag{3.45}$$

We see that *the induced worldsheet metric is conformal to the 2-dimensional Minkowski metric*; this is one reason for the name "conformal gauge".

3.3.1 Conformal gauge action

The conformal gauge $\lambda^+ = \lambda^- = 1$ is equivalent to

$$e = \frac{1}{T} \, , \qquad u = 0 \, . \tag{3.46}$$

Using this in the action we get the conformal gauge action, in phase-space form:

$$I[X, P] = \int dt \oint d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2T} P^2 - \frac{T}{2} (X')^2 \right\} \, . \tag{3.47}$$

The P equation of motion now simplifies to

$$P = T\dot{X} \, . \tag{3.48}$$

Using this to eliminate P, we arrive at the configuration-space form of the conformal gauge action

$$I[X] = \frac{T}{2} \int dt \oint d\sigma \left\{ \dot{X}^2 - (X')^2 \right\} = -\frac{T}{2} \int d^2\sigma \, \eta^{\mu\nu} \partial_\mu X \cdot \partial_\nu X \, . \tag{3.49}$$

This looks like a 2D Minkowski-space free field theory for D scalar fields X^m except that the X^0 term has the "wrong sign" for positive energy.

What about the Hamiltonian constraints? They appear to have disappeared now that we have set $\lambda^+ = \lambda^- = 1$; more on this later. For now let's just observe that if we set $P = \dot{X}$ in the constraints $\mathcal{H}_\pm = 0$ we get

$$\dot{X}^2 + (X')^2 = 0 \qquad \& \qquad \dot{X} \cdot X' = 0 . \tag{3.50}$$

These are precisely the equations (3.44) that we obtained previously using the equations of motion in conformal gauge. In fact, these are the equations that tell us that the induced metric is conformally flat, so they are a consequence of the equations of motion and the conformal gauge condition.

3.4 Polyakov action

Some aspects of the conformal gauge choice are simplified if we start from the Polyakov action[3]

$$I[X, \gamma] = -\frac{T}{2} \int d^2\sigma \sqrt{-\det \gamma}\, \gamma^{\mu\nu} g_{\mu\nu} , \tag{3.51}$$

where γ is a new *independent* worldsheet metric. As before, $g_{\mu\nu} = \partial_\mu X \cdot \partial_\nu X$ is the induced metric. The action depends on the metric γ only through its conformal class; in other words, given an everywhere non-zero function Ω, a rescaling

$$\gamma_{\mu\nu} \to \Omega^2 \gamma_{\mu\nu} \tag{3.52}$$

has no effect; the action is "Weyl invariant". This is easily seen since

$$\gamma^{\mu\nu} \to \Omega^{-2} \gamma^{\mu\nu} , \qquad \sqrt{-\det \gamma} \to \Omega^2 \sqrt{-\det \gamma} , \tag{3.53}$$

so the factors of Ω cancel from $\sqrt{-\det \gamma}\, \gamma^{\mu\nu}$.

Varying the Polyakov action with respect to $\gamma^{\mu\nu}$ we get the equation

$$g_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} \left(\gamma^{\rho\sigma} g_{\rho\sigma} \right) = 0 , \tag{3.54}$$

which we can rewrite as

$$\gamma_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \left(\Omega^{-2} = \frac{1}{2}\gamma^{\mu\nu} g_{\mu\nu} \right) . \tag{3.55}$$

In other words, the equation of motion for $\gamma_{\mu\nu}$ sets it equal to the induced metric $g_{\mu\nu}$ up to an irrelevant conformal factor. Back substitution gives us

$$I \to -\frac{T}{2} \int d^2\sigma \sqrt{-\det g}\, g^{\mu\nu} g_{\mu\nu} = -T \int d^2\sigma \sqrt{-\det g} , \tag{3.56}$$

so the Polyakov action is equivalent to the NG action.

Varying the Polyakov action with respect to X, we find the equation of motion

$$\partial_\mu \left(\sqrt{-\det \gamma}\, \gamma^{\mu\nu} \partial_\nu X^m \right) = 0 . \tag{3.57}$$

On substituting for γ using (3.55), we recover the NG equation of motion (3.5).

---

[3]The Polyakov action was actually introduced by Brink, DiVecchia and Howe, and by Deser and Zumino. Polyakov used it in the context of a path-integral quantization of the NG string, which we will consider later.

3.4.1 Relation to phase-space action

The Polyakov action can be found directly from the phase-space form of the action (3.15). Recall that elimination of P, using $P = e^{-1}D_t X$, gives

$$
\begin{aligned}
I &= \frac{1}{2} \int d^2\sigma \left\{ e^{-1} (D_t X)^2 - e\, (TX')^2 \right\} \qquad \left(D_t = \dot{X} - uX' \right) \\
&= \frac{1}{2} \int d^2\sigma \left\{ e^{-1}\dot{X}^2 - 2e^{-1}u\dot{X}\cdot X' + e^{-1}\left(u^2 - T^2 e^2\right)(X')^2 \right\}.
\end{aligned}
\tag{3.58}
$$

This has the Polyakov form

$$
I = -\frac{T}{2} \int d^2\sigma \sqrt{-\det\gamma}\;\gamma^{\mu\nu}\partial_\mu X \cdot \partial_\nu X\,,
\tag{3.59}
$$

with

$$
\sqrt{-\det\gamma}\,\gamma^{\mu\nu} = \frac{1}{Te}\begin{pmatrix} -1 & u \\ u & T^2 e^2 - u^2 \end{pmatrix}.
\tag{3.60}
$$

This tells us that

$$
\gamma_{\mu\nu} = \Omega^2 \begin{pmatrix} u^2 - T^2 e^2 & u \\ u & 1 \end{pmatrix},
\tag{3.61}
$$

for some irrelevant conformal factor Ω. Equivalently

$$
\begin{aligned}
ds^2(\gamma) &= \Omega^2 \left[\left(u^2 - T^2 e^2\right)dt^2 + 2u\,dt\,d\sigma + d\sigma^2 \right] \\
&= \Omega^2 \left(d\sigma + \lambda^+ dt \right)\left(d\sigma - \lambda^- dt \right),
\end{aligned}
\tag{3.62}
$$

where

$$
\lambda^\pm = Te \pm u\,.
\tag{3.63}
$$

You should recognise these as the Lagrange multipliers of the phase-space action in the form (3.22). These Lagrange multipliers become the two independent components of the conformal class of the Polyakov metric; i.e. they determine the metric $\gamma_{\mu\nu}$ up to multiplication by a conformal factor.

3.4.2 Conformal gauge redux

In the conformal gauge, $\lambda^+ = \lambda^- = 1$, so

$$
ds^2(\gamma) = \Omega^2 \left(-dt^2 + d\sigma^2 \right).
\tag{3.64}
$$

In other words, in the conformal gauge

$$
\gamma_{\mu\nu} = \Omega^2\,\eta_{\mu\nu}
\tag{3.65}
$$

for some non-zero function Ω. This is often presented as the definition of conformal gauge. A mathematical theorem states that for any 2-dimensional manifold with

metric γ, there exist *local* coordinates σ^μ such the metric is conformally flat, i.e. of the above form for Lorentzian signature.

Having chosen the conformal gauge, the equations of motion then tell us that the induced metric g is also conformally flat. We have already derived this result, but it now follows directly from (3.55) if we take (3.65) as the definition of conformal gauge.

Also, if we use the conformal gauge choice (3.65) in the Polyakov action we arrive directly at the configuration-space form of the conformal gauge action (3.49).

3.4.3 Residual gauge invariance

A feature of the conformal gauge choice is that it does not completely fix the gauge; there is a *residual* gauge invariance. In the conformal gauge the gauge transformations of $\lambda^\pm$ are

$$\delta\lambda^- = \dot{\xi}^- + \left(\xi^-\right)' = \sqrt{2}\,\partial_+\xi^-$$
$$\delta\lambda^+ = \dot{\xi}^+ - \left(\xi^+\right)' = -\sqrt{2}\,\partial_-\xi^+\,, \tag{3.66}$$

where $\partial_\pm$ are partial derivatives with respect to the light-cone worldsheet coordinates

$$\sigma^\pm = \frac{1}{\sqrt{2}}\left(\sigma \pm t\right)\,. \tag{3.67}$$

We see that $\delta\lambda^- = 0$ and $\delta\lambda^+ = 0$ if

$$\partial_\pm\xi^\mp = 0 \qquad \left[\Rightarrow \quad \xi^\pm = \xi^\pm(\sigma^\pm)\right]\,. \tag{3.68}$$

In other words the conformal gauge conditions are preserved by those gauge transformations for which ξ^+ is a function only of σ^+, and ξ^- is a function only of σ^-. As we are now going to see, these are **conformal transformations** of 2D Minkowski space.

Using the fact that $P = T\dot{X}$ in conformal gauge, the canonical gauge transformation of X becomes

$$\delta_\xi X = \frac{1}{2}\xi^-\left(\dot{X} - X'\right) + \frac{1}{2}\xi^+\left(\dot{X} + X'\right)$$
$$= \frac{1}{\sqrt{2}}\left(\xi^+\partial_+ - \xi^-\partial_-\right)X$$
$$= \zeta^\mu\partial_\mu X\,, \qquad \left[\zeta^\pm = \pm\xi^\pm/\sqrt{2}\right] \tag{3.69}$$

which is a Diff_2 transformation with vector field ζ. The Diff_2 transformation of the induced metric is[4]

$$\delta_\zeta g_{\mu\nu} = \left(\mathcal{L}_\zeta g\right)_{\mu\nu} \equiv \zeta^\rho\partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\zeta^\rho g_{\nu)\rho}\,. \tag{3.70}$$

---

[4]$\mathcal{L}_\zeta$ is the Lie derivative with respect to the vector field ζ.

The conformal group is the subgroup of the Diff_2 group generated by conformal Killing vector fields; these are such that

$$(\mathcal{L}_\zeta g)_{\mu\nu} = \chi g_{\mu\nu} \tag{3.71}$$

for some function χ. A solution of this equation with $\chi = 0$ is a Killing vector field; these are special cases of conformal Killing vector fields.

For n-dimensional Minkowski space-time with metric η, a conformal Killing vector field ζ is a solution to[5]

$$2\partial_{(\mu}\zeta^\rho \eta_{\nu)\rho} = \chi\,\eta_{\mu\nu}\,, \tag{3.72}$$

for some function χ. The number of linearly independent conformal Killing vector fields is finite for $n > 2$. Let's now consider the $n = 2$ case: in light-cone coordinates $\sigma^\pm$, for which $\eta_{+-} = 1$ and $\eta_{++} = \eta_{--} = 0$, the above condition can be written as

$$\begin{pmatrix} 2\partial_+\zeta^- & \partial_+\zeta^+ + \partial_-\zeta^- \\ \partial_+\zeta^+ + \partial_-\zeta^- & 2\partial_-\zeta^+ \end{pmatrix} = \begin{pmatrix} 0 & \chi \\ \chi & 0 \end{pmatrix}. \tag{3.73}$$

This equation determines the function χ but it also restricts $\zeta^\pm$ to satisfy $\partial_\mp\zeta^\pm = 0$. Since $\zeta^\pm \propto \xi^\pm$, this is equivalent to $\partial_\pm\xi^\mp = 0$, which is precisely (3.68).

3.4.4 The conformal algebra

The field equations of the conformal gauge action (3.49) are

$$\Box_2 X^m = 0\,, \qquad \Box_2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{3.74}$$

This is also what you get by setting $g_{\mu\nu} = \Omega^2\,\eta_{\mu\nu}$ in the NG equation of motion (3.5) because the conformal factor Ω^2 drops out. In worldsheet light-cone coordinates $\sigma^\pm$, the field equations become

$$\partial_+\partial_- X = 0 \qquad \Rightarrow \quad X = X_L(\sigma^+) + X_R(\sigma^-)\,. \tag{3.75}$$

As X_R (X_L) is constant for constant σ^- (σ^+) it describes a string profile that moves to the right (left) at the speed of light.

Now consider the effect of the residual gauge transformations on this solution:

$$\delta_\xi\left[X_L(\sigma^+) + X_R(\sigma^-)\right] = \frac{1}{\sqrt{2}}\left[\xi^+(\sigma^+)\partial_+ X_L(\sigma^+) - \xi^-(\sigma^-)\partial_- X_R(\sigma^-)\right] \tag{3.76}$$

Putting all functions of σ^+ on one side of this equation and all functions of σ^- on the other side we deduce that[6]

$$\delta_\xi X_L = \frac{1}{\sqrt{2}}\xi^+\partial_+ X_L\,, \qquad \delta_\xi X_R = -\frac{1}{\sqrt{2}}\xi^-\partial_- X_R\,. \tag{3.77}$$

---

[5]The round brackets indicate symmetrisation.

[6]Not quite because we could add a constant to $\delta_\xi X_L$ and subtract the same constant from $\delta_\xi X_R$, but this possibility only arises because of a similar ambiguity in the definition of the functions X_L and X_R, so the constant has no physical significance and we can set it to zero.

This shows that the algebra of the residual gauge transformations, is

$$\text{Diff}_1 \oplus \text{Diff}_1 . \tag{3.78}$$

Notice that this is the same as the algebra of canonical gauge transformations that we have *before* gauge fixing.

Elements of the algebra Diff_1 are vector fields on a one-dimensional manifold that are non-singular in a neighbourhood of the (arbitrarily chosen) origin. Let z be the coordinate (which could be either σ^+ or σ^-). Then a basis for Diff_1 is

$$\{z^n \partial_z \ ; \ n = 0, 1, 2, \ldots\} . \tag{3.79}$$

Any element of Diff_1 can be written as a linear combination:

$$\left(\xi_0 + \xi_1 z + \frac{1}{2} \xi_2 z^2 + \ldots \right) \partial_z = \xi(z) \partial_z . \tag{3.80}$$

This accords with the fact that the parameters $\xi^{\pm}$ of the residual gauge invariance in conformal gauge are functions only of $\sigma^{\pm}$. However, it is the infinite set of *constants* $\{\xi_n^{\pm}; n = 0, 1, 2 \ldots\}$ that should be regarded as the parameters of the residual gauge transformation with algebra $\text{Diff}_1 \oplus \text{Diff}_1$.

Suppose that we now allow these constant parameters to be arbitrary functions of time; e.g. for one

$$\xi^{\pm}(\sigma^{\pm}) \mapsto \xi^{\pm}(\sigma^{\pm}, t) = \xi_0^{\pm}(t) + \xi_1^{\pm}(t) \sigma^{\pm} + \frac{1}{2} \xi_2^{\pm}(t)(\sigma^{\pm})^2 + \ldots \tag{3.81}$$

Now both ξ^+ and ξ^- are arbitrary function of both σ and t, which is exactly what the parameters were *before* gauge fixing. The algebra hasn't changed but the parameters of an element of this algebra are now arbitrary functions of the worldsheet time t. What we now have is a canonical gauge transformation.

Moral: *The NG action is a 2D gauge theory of the 2D conformal group*, for which the algebra is $\text{Diff}_1 \oplus \text{Diff}_1$. In conformal gauge there is a residual conformal symmetry, with the same algebra but the parameters are now constants. This is similar to what happens in YM gauge theory if you choose a Lorentz covariant gauge like $\partial \cdot A = 0$. The local gauge invariance is broken by this condition but there is still a manifest invariance under global YM group transformations, with constant parameters.

N.B. The algebra Diff_1 has a finite-dimensional subalgebra, for which a basis of vector fields is

$$\left\{ J_- = \partial_z , \quad J_3 = z \partial_z , \quad J_+ = \frac{1}{2} z^2 \partial_z \right\} . \tag{3.82}$$

The commutation relations of these vector fields are [<u>Exercise</u>: check this]

$$[J_3, J_{\pm}] = \pm J_{\pm} , \qquad [J_-, J_+] = J_3 . \tag{3.83}$$

This is the algebra of $Sl(2; \mathbb{R})$, so the finite dimensional conformal algebra is $Sl(2; \mathbb{R}) \oplus Sl(2; \mathbb{R}) \cong SO(2, 2)$. In n dimensions the conformal algebra is $SO(2, n)$; for example, for $D = 4$ it is $SO(2, 4)$.

3.4.5 Conformal symmetry of conformal gauge action

Because the residual parameters $\{\xi_n; n = 0, 1, 2 \ldots\}$ of a Diff_1 transformation are constants, it is tempting to think of the residual *gauge* transformation as a *symmetry*. This idea can't be right but let's see how far we can push it.

Start from the conformal gauge action in worldsheet light-cone coordinates

$$I[X] = -T \int d^2\sigma \, \partial_+ X \cdot \partial_- X \,. \tag{3.84}$$

Let's compute the variation that results from the transformation $\delta X = \frac{1}{\sqrt{2}} \xi^+ \partial_+ X$. We get

$$\delta_{\xi^+} I = -\frac{T}{\sqrt{2}} \int d^2\sigma \left\{ \partial_+ \left(\xi^+ \partial_+ X \right) \cdot \partial_- X + \partial_+ X \cdot \partial_- \left(\xi^+ \partial_+ X \right) \right\} \,. \tag{3.85}$$

Integrating by parts in the first term[7] we find that

$$\delta_{\xi^+} I = -\frac{T}{\sqrt{2}} \int d^2\sigma \, \partial_- \xi^+ \left(\partial_+ X \right)^2 \,. \tag{3.86}$$

A similar calculation gives

$$\delta_{\xi^-} I = \frac{T}{\sqrt{2}} \int d^2\sigma \, \partial_+ \xi^- \left(\partial_- X \right)^2 \,. \tag{3.87}$$

We thus confirm that the action is invariant if $\partial_\pm \xi^\mp = 0$.

What are the Noether charges? Let's consider the closed string, for which the parameters $\xi^\pm(\sigma^\pm)$ must be periodic in $\sigma^\pm$ with period $\sqrt{2}\,\pi$ (because $\sigma \sim \sigma + 2\pi$) so we can write them as Fourier series. Allowing for t-dependent Fourier coefficients, we have[8]

$$\xi^\pm = \sum_{n \in \mathbb{Z}} e^{\pm in\sqrt{2}\sigma^\pm} \xi_n^\pm(t) \,. \tag{3.88}$$

Substituting this in the above expressions for $\delta_{\xi^\pm} I$, we find that

$$\delta_{\xi^\pm} I = \int dt \sum_{n \in \mathbb{Z}} \dot{\xi}_n^\pm \sqrt{2} \oint d\sigma^\pm e^{\pm in\sqrt{2}\sigma^\pm} \Theta_{\pm\pm} \,, \tag{3.89}$$

where

$$\Theta_{++} = \frac{T}{2} (\partial_+ X)^2 \,, \qquad \Theta_{--} = \frac{T}{2} (\partial_- X)^2 \,. \tag{3.90}$$

These are the only non-zero components, in light-cone coordinates, of the stress-tensor

$$\Theta_{\mu\nu} = \frac{T}{2} \left[\partial_\mu X \cdot \partial_\nu X - \frac{1}{2} \eta_{\mu\nu} \left(\eta^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X \right) \right] \,. \tag{3.91}$$

---

[7]The only boundaries are at the initial and final times and we can ignore total time derivatives.

[8]The optional $\pm$ sign in the exponent is for agreement with later conventions.

In other words, the Noether charges are the Fourier coefficients of the light-cone components of the stress tensor for the conformal gauge action (with the Fourier expansion taken in terms of the variable on which the stress tensor component depends). They can also be written as

$$\bar{L}_n^\pm = e^{int} \oint d\sigma \, e^{\pm in\sigma} \Theta_{\pm\pm} \,. \tag{3.92}$$

Notice, in particular, that the Noether charges are all zero when $\Theta_{++} = \Theta_{--} = 0$, and *vice versa*. You can verify their time-independent, as a consequence of the equation of motion $\Box_2 X = 0$, by using the identity

$$\dot{\Theta}_{\pm\pm} \equiv \frac{T}{\sqrt{2}} \, \partial_\pm X \cdot \Box_2 X \, \pm \, \Theta'_{\pm\pm} \,. \tag{3.93}$$

We have now confirmed the conformal invariance of the conformal gauge action (3.84). Given only this action, there would be no reason not to say that its conformal invariance is a symmetry. However, we know that the Hamiltonian constraints in conformal gauge are $(\dot{X} \pm X')^2 = 0$, for either sign. In worldsheet light-cone coordinates $\sigma^\pm$, these constraints become

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \qquad \Rightarrow \qquad \Theta_{++} = \Theta_{--} = 0 \,. \tag{3.94}$$

So, if we take into account the string origin of the conformal gauge action, the Noether charges of the residual conformal "symmetry" are constrained to vanish. This is a consequence of the fact that the residual conformal "symmetry" of the conformal gauge action is actually a residual *gauge invariance* of the NG string in conformal gauge.

But there is still a puzzle. Given *only* the conformal gauge action (3.84), how would we know that its origin was in the NG string? How would we know that we should impose the conditions $\Theta_{++} = \Theta_{--} = 0$? It appears that when we set $\lambda^+ = \lambda^- = 1$ in the action we lose the information previously provided by varying λ^+ and λ^-, i.e. the constraints. If you look at string theory texts you will often find a statement to this effect. However, *it is impossible to "lose" information by imposing a valid gauge condition because, by definition, this removes only redundancy.*

There must be something wrong with the conformal gauge condition. To see what is going on here, we return to the point particle.

3.4.6 "Conformal gauge" for the particle

The particle analog of the conformal gauge is

$$e = 1/m \,. \tag{3.95}$$

Substituting this into the phase-space action we find that

$$I \; \to \; \int dt \left\{ \dot{x}^m p_m - \frac{p^2}{2m} - \frac{m}{2} \right\} . \tag{3.96}$$

Eliminating p by its equation of motion $p = m\dot{x}$ we arrive at the action

$$I[x] = \frac{m}{2} \int dt \left\{ \dot{x}^2 - 1 \right\} . \tag{3.97}$$

This is the particle analog of the conformal gauge action. The equation of motion is

$$\ddot{x} = 0 , \tag{3.98}$$

which is the "1D wave equation". That's all we get from $I[x]$ but if we recall that the equations of motion, prior to gauge fixing, imply that $me = \sqrt{-\dot{x}^2}$, which becomes $1 = \sqrt{-\dot{x}^2}$ when $e = 1/m$, we see that the $e = 1/m$ choice of gauge is equivalent, using the equations of motion, to the constraint

$$\dot{x}^2 + 1 = 0 . \tag{3.99}$$

This is also what you get by using $p = m\dot{x}$ in the mass-shell constraint. We appear to have lost this constraint from the gauge-fixed action.

Before addressing this problem, let's notice that the gauge choice (3.95) does not completely fix the gauge. The canonical gauge transformation of e is $\delta e = \dot{\alpha}$, so the gauge choice is preserved by a gauge transformation with parameter $\alpha = \bar{\alpha}$, a *constant*. Let's check that the "conformal gauge" action (3.97) is invariant for constant α. For arbitrary $\alpha(t)$ we have

$$\delta x = m \alpha \dot{x} \quad \Rightarrow \quad \delta I[X] = \frac{m^2}{2} \int_{t_A}^{t_B} dt \, \dot{\alpha} \left(\dot{x}^2 + 1 \right) + \frac{m^2}{2} \left[\alpha \left(\dot{x}^2 - 1 \right) \right]_{t_A}^{t_B} . \tag{3.100}$$

Apart from the boundary term, this is zero for $\alpha = \bar{\alpha}$. We also see that the Noether charge of this residual symmetry is[9]

$$Q \propto \dot{x}^2 + 1 . \tag{3.101}$$

But the constraint is precisely $Q = 0$, showing that the apparent residual symmetry is really a gauge invariance.

However, we still have to explain how the constraint can be "lost" when the gauge condition (3.95) is imposed. The reason is simple: we can't impose this condition. To see why, observe that

$$\delta_\alpha \left[m \int_{t_A}^{t_B} dt \, e \right] = m [\alpha]_{t_A}^{t_B} = 0 , \tag{3.102}$$

---

[9]There is an ambiguity due to the fact that m is trivially a separate constant of the motion; the ambiguity has been resolved so as to make this example as much as possible like the string case.

where the last equality is due to the fact that $\alpha(t)$ must be zero at the initial and final times; otherwise, the action prior to gauge fixing is not gauge invariant. The integral is the lapsed proper time (using equation $me = \sqrt{-\dot{x}^2}$). This is gauge invariant and *cannot be changed by a gauge transformation*, so setting e to any particular constant fixes a gauge-invariant quantity; this is more than just fixing the gauge. The best that we can do is to set

$$e = s\,, \tag{3.103}$$

for variable constant s, on which the gauge-fixed action still depends. Varying the gauge-fixed phase-space action with respect to s yields the *integrated constraint*

$$\int_{t_A}^{t_B} dt \left(p^2 + m^2 \right) = 0\,. \tag{3.104}$$

However, since $p^2 + m^2$ is a constant of the motion, this is equivalent, when combined with the equation of motion $\dot{p} = 0$ to the unintegrated constraint $p^2 + m^2 = 0$, which we can think of as a *initial condition*. This interpretation would be obvious if, instead of choosing $e = s$, we choose $e = 1/m$ *almost everywhere*, leaving it free in a neighbourhood of the initial time.

The puzzle of the "lost" constraints for the string in conformal gauge has essentially the same resolution. Recall that a theorem tells us that we can choose local worldsheet coordinates such that the worldsheet metric is conformally flat. This theorem does not tell us that we can do this globally. In the case of a cylindrical worldsheet, one can find conformal coordinates σ^μ almost everywhere; in particular, everywhere outside a neighbourhood of the initial time. There then remain variables in the action whose variation implies the constraints $\Theta_{++} = \Theta_{--} = 0$ at the initial time. The identity (3.93) shows that if $\Theta_{\pm\pm} = 0$ initially, *for all σ*, then it will be zero at all later times as a consequence of the equation of motion $\Box_2 X = 0$.

Moral. The string origin of the conformal gauge action implies that the Hamiltonian constraints appear as constraints on initial conditions.

3.5 Solving the NG equations in conformal gauge

Locally, the NG equations reduce to the 2D wave equation in conformal gauge, which is easily solved. In particular, we can solve the 2D wave-equation for X^0 by setting $X^0(t,\sigma) = t$. In fact, we can use the residual conformal symmetry in conformal gauge to set $X^0 = t$ without loss of generality.

Let's check this. Recall that the residual conformal transformation of X^0 is

$$\delta X^0 = \frac{1}{\sqrt{2}} \left[\xi^+(\sigma^+)\partial_+ X^0 - \xi^-(\sigma^-)\partial_- X^0 \right]\,. \tag{3.105}$$

Setting $X^- = t$, we get

$$0 = \xi^+(\sigma^+) + \xi^-(\sigma^-) \quad \Rightarrow \quad \xi^\pm = \pm\bar{\xi}\,, \tag{3.106}$$

for some *constant* $\bar{\xi}$. The only surviving part of the conformal transformation is therefore

$$\delta_{\bar{\xi}} X = \frac{1}{\sqrt{2}} \bar{\xi} \left(\partial_+ + \partial_-\right) X = \bar{\xi} X'. \tag{3.107}$$

This is how X transforms due to a *constant* shift of σ ($\sigma \to \sigma - \bar{\xi}$); i.e. a change of origin for the string coordinate σ.

We can easily write down the general solution for $\vec{X}$ that satisfies the 2D wave equation but to have a solution of the NG equations we must also solve the conformal gauge constraints, which are now

$$2 \left| \partial_\pm \vec{X} \right|^2 = 1. \tag{3.108}$$

Given a solution of the 2D wave equation for $\vec{X}$ we may check directly to see whether the constraints are satisfied. Alternatively, we may compute the induced metric to see if it is conformally flat; if it is then the conformal gauge constraints will be satisfied because they are precisely the conditions for conformal flatness of the induced metric.

Let's apply these ideas to the closed string configuration in a 5-dimensional space-time with $X^0 = t$ and

$$Z \equiv X^1 + iX^2 = \frac{1}{2n} e^{in(\sigma-t)}, \qquad W \equiv X^3 + iX^4 = \frac{1}{2m} e^{im(\sigma+t)}. \tag{3.109}$$

This configuration clearly solves the 2D wave equation. If the induced metric is conformally flat then it will also solve the full NG equations, including the constraints. A calculation gives

$$\begin{aligned}
ds^2 \big|_{\text{ind}} &= -\left(dX^0\right)^2 + |dZ|^2 + |dW|^2 \\
&= -dt^2 + \frac{1}{4} \left(d\sigma - dt\right)^2 + \frac{1}{4} \left(d\sigma + dt\right)^2 \\
&= \frac{1}{2} \left(-dt^2 + d\sigma^2\right).
\end{aligned} \tag{3.110}$$

In other words,

$$g_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu}. \tag{3.111}$$

This is flat, and hence conformally flat, so the given configuration is a solution of the NG equations. [**N.B.** It is not necessary to also check (3.108) because this is a consequence of conformal flatness, but it is a good idea to check it.]

This solution has the special property of being stationary; the string is motionless in a particular rotating frame. To see this, we first compute the proper length L of the string. Setting $t = t_0$ in the induced worldsheet metric (for some constant t_0) we see that $d\ell^2 = \frac{1}{2} d\sigma^2$, so

$$L = \frac{1}{\sqrt{2}} \oint d\sigma = \sqrt{2}\, \pi. \tag{3.112}$$

It is rather surprising that this should be constant, i.e. independent of t_0; it means that the motion of the string is supporting it against collapse due to its tension. To check this, we may compute the total energy, which is

$$H = \oint d\sigma P^0 = T \oint d\sigma \dot{X}^0 = 2\pi T \,. \tag{3.113}$$

We see that

$$H = \sqrt{2}\, TL = TL + \left(\sqrt{2} - 1\right) TL \tag{3.114}$$

The first term is the potential energy of the string. The second term is therefore kinetic energy. The string is supported against collapse by rotation in the Z and W planes. The string is circular for $n = m$, and planar, so a circular planar loop of string can be supported against collapse by rotation in two orthogonal planes provided that neither of them coincides with the plane of the string loop.

3.6 Open string boundary conditions

An open string has two ends. We shall choose the parameter length to be π, so the action in Hamiltonian form is

$$I = \int dt \int_0^\pi d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left[P^2 + (TX')^2 \right] - uX' \cdot P \right\} \,. \tag{3.115}$$

What are the possible boundary conditions at the ends of the string?

Principle: the action should be stationary when the equations of motion are satisfied. In other words, when we vary the action to get the equations of motion, the boundary terms arising from integration by parts must be zero; otherwise the functional derivative of the action is not defined.

Applying this principle to the above action, we see that boundary terms can arise only when we vary X' and integrate by parts to get the derivative (with respect to σ) off the δX variation (we can ignore any boundary terms in time). These boundary terms are

$$\delta I|_{\text{on-shell}} = - \int dt \left[(T^2 eX' + uP) \cdot \delta X \right]_{\sigma=0}^{\sigma=\pi} \,. \tag{3.116}$$

Here, "on-shell" is shorthand for "using the equations of motion". [Exercise: check this].

It would make no physical sense to fix X^0 at the endpoints, and if X^0 is free then so is $\dot{X}^0$ and hence P^0 when we use the equations of motion, so the boundary term with the factor of δX^0 will be zero only if we impose the conditions

$$u|_{ends} = 0 \,, \qquad \left(X^0\right)'\Big|_{ends} = 0 \,. \tag{3.117}$$

Given that $e \neq 0$ (we'll pass over this point) we conclude that

$$\left. \vec{X}' \cdot \delta \vec{X} \right|_{ends} = 0 \,. \tag{3.118}$$

What this means is that *at each end* we may *choose cartesian space coordinates* $\vec{X}$ such that *for any given component*, call it X_*, we have

$$\textbf{either} \quad X'_* = 0 \qquad \text{(Neumann b.c.s)} \,,$$
$$\textbf{or} \quad \delta X_* = 0 \quad \Rightarrow \quad X_* = \bar{X}_* \text{ (constant)} \qquad \text{(Dirichlet b.c.s)} \tag{3.119}$$

There are many possibilities. The only one that does not break Lorentz invariance is *free-end* boundary conditions

$$\left. X' \right|_{ends} = 0 \,. \tag{3.120}$$

This implies that $(X')^2$ is zero at the ends of the string. The open string mass-shell constraint then implies that P^2 is zero at the endpoints, and since

$$\left. P \right|_{ends} = e^{-1} \left. \left(\dot{X} - uX' \right) \right|_{ends} = e^{-1} \left. \dot{X} \right|_{ends} = 0 \,, \tag{3.121}$$

we deduce that $\dot{X}^2$ is zero at the ends of the string; i.e. *the string endpoints move at the speed of light.*

3.7 Fourier expansion: closed string

The worldsheet fields of the closed string are periodic functions of σ with (by convention) period 2π, so we can express them as Fourier series. It is convenient to express (X, P) as Fourier series by starting with the combinations $P \pm TX'$ (because the gauge transformations act separately on $P + TX'$ and $P - TX'$), so we write

$$P - TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{ik\sigma} \alpha_k(t) \qquad (\alpha_{-k} = \alpha_k^*)$$

$$P + TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{-ik\sigma} \tilde{\alpha}_k(t) \qquad (\tilde{\alpha}_{-k} = \tilde{\alpha}_k^*) \,. \tag{3.122}$$

Recall that worldsheet parity $\sigma \to -\sigma$ exchanges $P + TX'$ with $P - TX'$. Because of the relative minus sign in the exponent of the Fourier series, this means that worldsheet parity exchanges α_k with $\tilde{\alpha}_k$:

$$\alpha_k \quad \leftrightarrow \quad \tilde{\alpha}_k \,. \tag{3.123}$$

We can integrate either of the above equations to determine the total D-momentum in terms of Fourier modes, since X' integrates to zero for a closed string; this gives us

$$p = \oint P d\sigma = \begin{cases} \sqrt{4\pi T} \alpha_0 \\ \sqrt{4\pi T} \tilde{\alpha}_0 \end{cases} \quad \Rightarrow \quad \alpha_0 = \tilde{\alpha}_0 = \frac{p}{\sqrt{4\pi T}} \,. \tag{3.124}$$

By adding the Fourier series expressions for $P \pm TX'$ we now get

$$P(t,\sigma) = \frac{p(t)}{2\pi} + \sqrt{\frac{T}{4\pi}} \sum_{k \neq 0} e^{ik\sigma} [\alpha_k(t) + \tilde{\alpha}_{-k}(t)] \; . \tag{3.125}$$

By subtracting we get

$$X' = -\frac{1}{\sqrt{4\pi T}} \sum_{k \neq 0} e^{ik\sigma} [\alpha_k(t) - \tilde{\alpha}_{-k}(t)] \; , \tag{3.126}$$

which we may integrate to get the Fourier series expansion for X:

$$X(t,\sigma) = x(t) + \frac{1}{\sqrt{4\pi T}} \sum_{k \neq 0} \frac{i}{k} e^{ik\sigma} [\alpha_k(t) - \tilde{\alpha}_{-k}(t)] \; . \tag{3.127}$$

The integration constant (actually a function of t) can be interpreted as the position of the centre of mass of the string; we should expect it to behave like a free particle.

Using the Fourier series expansions of (X, P) we now find that

$$\oint d\sigma \, \dot{X}^m P_m = \dot{x}^m p_m + \sum_{k=1} \frac{i}{k} \left(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k} \right) + \frac{d}{dt} () \tag{3.128}$$

Exercise: check this [*Hint.* Cross terms that mix α with $\tilde{\alpha}$ are all in the total time derivative term, and the $k < 0$ terms in the resulting sum double the $k > 0$ terms].

Next we Fourier expand the constraint functions $\mathcal{H}_\pm$:

$$\mathcal{H}_- = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\sigma} L_n \, , \qquad \mathcal{H}_+ = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\sigma} \tilde{L}_n \, . \tag{3.129}$$

Inverting to get the Fourier coefficients in terms of the functions $\mathcal{H}_\pm$, we get

$$L_n = \oint d\sigma \, e^{-in\sigma} \mathcal{H}_- = \frac{1}{4T} \oint d\sigma \, e^{-in\sigma} \, (P - TX')^2$$

$$\tilde{L}_n = \oint d\sigma \, e^{in\sigma} \mathcal{H}_+ = \frac{1}{4T} \oint d\sigma \, e^{in\sigma} \, (P + TX')^2 \; . \tag{3.130}$$

Inserting the Fourier expansions (3.122) we find that (Exercise: check this)

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \cdot \alpha_{n-k} \, , \qquad \tilde{L}_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \tilde{\alpha}_k \cdot \tilde{\alpha}_{n-k} \, . \tag{3.131}$$

We may similarly expand the Lagrange multipliers as Fourier series but it should be clear in advance that there will be one Fourier mode of λ^- for each L_n (let's call this λ_{-n}) and one Fourier mode of λ^+ for each $\tilde{L}_n$ (let's call this $\tilde{\lambda}_{-n}$). We may now write down the closed string action in terms of Fourier modes. It is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1} \frac{i}{k} \left(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k} \right) - \sum_{n \in \mathbb{Z}} \left(\lambda_{-n} L_n + \tilde{\lambda}_{-n} \tilde{L}_n \right) \right\} \; . \tag{3.132}$$

This action is manifestly Poincaré invariant. The Noether charges are

$$\mathcal{P}_m = \oint d\sigma P_m = p_m\,,$$

$$\mathcal{J}^{mn} = 2\oint d\sigma X^{[m}P^{n]} = 2x^{[m}p^{n]} + S^{mn}\,, \tag{3.133}$$

where the spin part of the Lorentz charge is (Exercise: check this)

$$S^{mn} = -2\sum_{k=1}^{\infty} \frac{i}{k}\left(\alpha_{-k}^{[m}\alpha_k^{n]} + \tilde{\alpha}_{-k}^{[m}\tilde{\alpha}_k^{n]}\right)\,. \tag{3.134}$$

- **Lemma**. For a Lagrangian of the form

$$L = \frac{i}{c}\dot{\alpha}\alpha^* - H(\alpha, \alpha^*) \tag{3.135}$$

  for constant c, the PB of the canonical variables takes the form

$$\{\alpha, \alpha^*\}_{PB} = -ic\,. \tag{3.136}$$

  To see this set $\alpha = \sqrt{|c|/2}\,[q + i\,\mathrm{sign}(c)p]$ to get $L = \dot{q}p - H$, for which we know that $\{q, p\}_{PB} = 1$. This implies the above PB for α and α^*.

Using this lemma we may read off from the action that the non-zero Poisson brackets of canonical variables are $\{x^m, p_n\}_{PB} = \delta_n^m$ and

$$\left\{\alpha_k^m, \alpha_{-k}^n\right\}_{PB} = -ik\eta^{mn}\,, \quad \left\{\tilde{\alpha}_k^m, \tilde{\alpha}_{-k}^n\right\}_{PB} = -ik\eta^{mn}\,. \tag{3.137}$$

Using these PBs, and the Fourier series expressions for (X, P), we may compute the PB of $X(\sigma)$ with $P(\sigma')$. [Exercise: check that the result agrees with (3.23).]

We may also use the PBs (3.137) to compute the PBs of the constraint functions $(L_n, \tilde{L}_n)$. The non-zero PBs are (Exercise: check this)

$$\{L_k, L_j\}_{PB} = -i\,(k-j)\,L_{k+j}\,, \qquad \left\{\tilde{L}_k, \tilde{L}_j\right\}_{PB} = -i\,(k-j)\,\tilde{L}_{k+j}\,. \tag{3.138}$$

We may draw a number of conclusions from this result:

- The constraints are first class, so the L_n and $\tilde{L}_n$ generate gauge transformations, for each $n \in \mathbb{Z}$.

- The structure functions of the algebra of first-class constraints are *constants*. This means that the $(L_n, \tilde{L}_n)$ span an infinite dimensional *Lie algebra*.

- The Lie algebra of the gauge group is a direct sum of two copies of the same algebra, sometimes called the Witt algebra.

The Witt algebra is also the algebra of diffeomorphisms of the circle. Suppose we have a circle parameterized by $\theta \sim \theta + 2\pi$ (we could take θ to be σ^+ or σ^-). The algebra Diff_1 of diffeomorphisms is spanned by the vector fields on the circle, and since these are periodic we may take as a basis set the vector fields $\{V_n; n \in \mathbb{Z}\}$, where

$$V_n = e^{in\theta} \frac{d}{d\theta} . \tag{3.139}$$

The commutator of two basis vector fields is

$$[V_k, V_j] = -i(k - j) V_{k+j} . \tag{3.140}$$

Corollary: the algebra of the gauge group is $\text{Diff}_1 \oplus \text{Diff}_1$, as claimed earlier.

3.8 Fourier expansion: open string

The open string has two ends. We will choose the ends to be at $\sigma = 0$ and $\sigma = \pi$, so the parameter length of the string is π (this is just a convention). We shall first consider the case of free-end (Neumann) boundary conditions. Then we shall go on to see how the results change when the string ends are not free to move in certain directions (mixed Neumann/Dirichlet b.c.s).

3.8.1 Free-ends

If the ends of the string are free, we must require X' to be zero at the ends, i.e. at $\sigma = 0$ and $\sigma = \pi$. We shall proceed in a way that will allow us to take over results from the closed string; we shall use a "doubling trick":

- First, we extend the definition of (X, P) from the interval $[0, \pi]$ to the interval $[0, 2\pi]$ in such a way that (X, P) are periodic on this doubled interval. This will allow us to use the closed string Fourier series expressions.

- Next, we impose a condition that relates (X, P) in the interval $[\pi, 2\pi]$ to (X, P) in the interval $[0, \pi]$; this will ensure that any additional degrees of freedom that we have introduced by doubling the interval are removed. Notice that if $\sigma \in [0, \pi]$ then $-\sigma \sim -\sigma + 2\pi \in [\pi, 2\pi]$, so we need to relate the worldsheet fields at σ to their values at $-\sigma$. The condition that does this should be consistent with periodicity in the doubled interval, but it should also imply the free-end b.c.s at $\sigma = 0, \pi$.

The solution to these requirements is to impose the condition

$$(P + TX')(\sigma) = (P - TX')(-\sigma) . \tag{3.141}$$

This is consistent with periodicity, and setting $\sigma = 0$ it implies that $X'(0) = 0$. It also implies that $X'(\pi) = 0$ because $-\pi \sim \pi$ by periodicity. In terms of Fourier modes, the condition (3.141) becomes

$$\tilde{\alpha}_k = \alpha_k \qquad (k \in \mathbb{Z}). \tag{3.142}$$

Using this in (3.122) we have

$$P \pm TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{\mp ik\sigma} \alpha_k \qquad (\alpha_{-k} = \alpha_k^*) . \tag{3.143}$$

Equivalently,

$$P = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} \cos(k\sigma)\, \alpha_k , \qquad X' = -\frac{i}{\sqrt{\pi T}} \sum_{k \in \mathbb{Z}} \sin(k\sigma)\, \alpha_k . \tag{3.144}$$

Integrating to get X, and defining $p(t)$ by

$$\alpha_0 = \frac{p}{\sqrt{\pi T}} , \tag{3.145}$$

we have

$$X(t,\sigma) = x(t) + \frac{1}{\sqrt{\pi T}} \sum_{k \neq 0} \frac{i}{k} \cos(k\sigma)\, \alpha_k ,$$

$$P(t,\sigma) = \frac{p(t)}{\pi} + \sqrt{\frac{T}{\pi}} \sum_{k \neq 0} \cos(k\sigma)\, \alpha_k . \tag{3.146}$$

Notice that p is again the total momentum since

$$\int_0^\pi d\sigma\, P(t,\sigma) = p , \tag{3.147}$$

but its relation to α_0 differs from that of the closed string.

Using the Fourier series expansions for (X, P) we find that

$$\int_0^\pi \dot{X}^m P_m \, d\sigma = \dot{x}^m p_m + \sum_{k=1}^\infty \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} + \frac{d}{dt}\, () . \tag{3.148}$$

Because of (3.141) we also have $\mathcal{H}_+(\sigma) = \mathcal{H}_-(-\sigma)$, so we should impose a similar relation on the Lagrange multipliers

$$\lambda^+(\sigma) = \lambda^-(-\sigma) \qquad (\Rightarrow \ u|_{\text{ends}} = 0 \ \ \& \ \ e'|_{\text{ends}} = 0) . \tag{3.149}$$

Then

$$\int_0^\pi d\sigma \left(\lambda^- \mathcal{H}_- + \lambda^+ \mathcal{H}_+ \right) = \int_0^\pi d\sigma\, \lambda^-(\sigma)\mathcal{H}_-(\sigma) + \int_0^\pi d\sigma\, \lambda^-(-\sigma)\mathcal{H}_-(-\sigma)$$

$$= \int_0^\pi d\sigma\, \lambda^-(\sigma)\mathcal{H}_-(\sigma) + \int_{-\pi+2\pi}^{0+2\pi} d\sigma\, \lambda^-(\sigma)\mathcal{H}_-(\sigma)$$

$$= \oint d\sigma\, \lambda^- \mathcal{H}_- , \tag{3.150}$$

and we can now use the Fourier series expansions of the closed string.

The final result for the open string action in Fourier modes is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n \in \mathbb{Z}} \lambda_{-n} L_n \right\} . \tag{3.151}$$

The difference with the closed string is that we have one set of oscillator variables instead of two. We can now read off the non-zero PB relations

$$\{x^m, p_n\}_{PB} = \delta^m_n , \qquad \{\alpha^m_k, \alpha^n_{-k}\}_{PB} = -ik\eta^{mn} , \tag{3.152}$$

and we can use this to show that

$$\{L_n, \alpha^m_k\}_{PB} = ik\alpha_{n+k} . \tag{3.153}$$

This means that the gauge variation of α_k is

$$\delta_\xi \alpha^m_k = \sum_{n \in \mathbb{Z}} \xi_{-n} \{\alpha^m_k, L_n\}_{PB} = -ik \sum_{n \in \mathbb{Z}} \xi_{-n} \alpha^m_{n+k} , \tag{3.154}$$

where ξ_n are parameters. To compute the gauge transformation of (x, p) we use the relation $p = \sqrt{\pi T}\, \alpha_0$ and the fact that

$$L_0 = \frac{1}{2}\alpha_0^2 + \dots , \qquad L_n = \alpha_0 \cdot \alpha_n + \dots , \tag{3.155}$$

where the dots indicate terms that do not involve α_0, to compute

$$\delta_\xi x^m = \frac{1}{\sqrt{\pi T}} \sum_{n \in \mathbb{Z}} \xi_{-n} \alpha_n , \qquad \delta p_m = 0 . \tag{3.156}$$

Finally, one may verify that the action is invariant if

$$\delta_\xi \lambda_n = \dot{\xi}_n + i \sum_{k \in \mathbb{Z}} (2k - n)\, \xi_k \lambda_{n-k} . \tag{3.157}$$

3.8.2 Parallel p-plane boundary conditions

Let's now suppose that we have mixed Neumann and Dirichlet boundary conditions. We will divide the cartesian coordinates X^m into two sets

$$\left\{ X^{\hat{m}} \, ; \hat{m} = 0, 1, \dots, p \right\} , \qquad \left\{ X^{\check{m}} \, ; \check{m} = p+1, \dots, D-1 \right\} . \tag{3.158}$$

We will suppose that $X^{\hat{m}}(t, \sigma)$ are subject to Neumann b.c.s. and that $X^{\check{m}}(t, \sigma)$ are subject to Dirichlet b.c.s, so the string is stretched between a p-plane at the origin and a parallel p-plane situated at $X^{\check{m}} = L^{\check{m}}$. The boundary conditions corresponding to this situation are

$$\left(X^{\hat{m}} \right)' \Big|_{ends} = 0 \qquad X^{\check{m}} \Big|_{\sigma=0} = 0 \quad \& \quad X^{\check{m}} \Big|_{\sigma=\pi} = L^{\check{m}} . \tag{3.159}$$

Notice that these boundary conditions break invariance under the $SO(1, D − 1)$ Lorentz group to invariance under the subgroup $SO(1, p) \times SO(D − p − 1)$. In particular the D-dimensional Lorentz invariance is broken to a $(p + 1)$-dimensional Lorentz invariance.

To get the Fourier series expansions of $P \pm TX'$ for these b.c.s. we may again use the doubling trick, but the constraint relating the components of $(P \pm TX')$ at σ to the components at $−\sigma$ now depends on whether it is a $\hat{m}$ component or a $\check{m}$ component. For the $\hat{m}$ components we choose the (3.141) condition, which implies that $(X^{\hat{m}})'$ is zero at the endpoints. For the $\check{m}$ components we impose the condition

$$(P + TX')^{\check{m}}(\sigma) = −(P − TX')^{\check{m}}(−\sigma), \tag{3.160}$$

which implies that $P_{\check{m}}$ is zero at the endpoints. In terms of the oscillator variables, the condition (3.160) becomes

$$\tilde{\alpha}_k^{\check{m}} = −\alpha_k^{\check{m}}, \tag{3.161}$$

which gives

$$(P \pm TX')^{\check{m}} = \mp\sqrt{\frac{T}{\pi}} \sum_{k\in\mathbb{Z}} e^{\mp ik\sigma} \alpha_k^{\check{m}}. \tag{3.162}$$

Taking the sum and the difference for the two signs, we find that

$$P^{\check{m}} = i\sqrt{\frac{T}{\pi}} \sum_{k\in\mathbb{Z}} \sin(k\sigma)\alpha_k^{\check{m}}, \tag{3.163}$$

which is indeed zero at $\sigma = 0, \pi$, and

$$\left(X^{\check{m}}\right)' = −\frac{\alpha_0^{\check{m}}}{\sqrt{\pi T}} − \frac{1}{\sqrt{\pi T}} \sum_{k\neq0} \cos(k\sigma)\alpha_k^{\check{m}}. \tag{3.164}$$

Integrating the latter equation over the string, we find that

$$L^{\check{m}} = \int_0^\pi d\sigma \left(X^{\check{m}}\right)' = −\sqrt{\pi}T\,\alpha_0^{\check{m}} \quad \Rightarrow \quad \alpha_0^{\check{m}} = −\sqrt{\frac{T}{\pi}}\,L^{\check{m}}, \tag{3.165}$$

and hence that

$$X^{\check{m}} = \frac{L^{\check{m}}\sigma}{\pi} − \frac{1}{\sqrt{\pi T}} \sum_{k\neq0} \frac{1}{k}\sin(k\sigma)\alpha_k^{\check{m}}, \tag{3.166}$$

which satisfies the boundary conditions (3.159).

Using the Fourier series expressions for $X^{\check{m}}$ and $P_{\check{m}}$, we find that (sum over $\check{m}$)

$$\int_0^\pi d\sigma\,\dot{X}^{\check{m}}P_{\check{m}} = \sum_{k=1}^\infty \frac{i}{k}\,\dot{\alpha}_k^{\check{m}}\alpha_{-k}^{\check{m}} + \frac{d}{dt}(). \tag{3.167}$$

Also, we still have $\mathcal{H}_+(\sigma) = \mathcal{H}_-(-\sigma)$, irrespective of whether the boundary conditions are Neumann or Dirichlet, so the Fourier mode expansion of the constraints is unchanged, and hence the full action in terms of Fourier modes is

$$I = \int dt \left\{ \dot{x}^{\hat{m}} p_{\hat{m}} + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n \in \mathbb{Z}} \lambda_{-n} L_n \right\} , \qquad (3.168)$$

where, as before, $L_n = \frac{1}{2} \sum_k \alpha_k \cdot \alpha_{n-k}$. Notice the *absence* of a $\dot{x}^{\check{m}} p_{\check{m}}$ term. Apart from this, the only difference to the free-end case is the changed significance of the components of α_0^m: we now have

$$\alpha_0^{\hat{m}} = \frac{p^{\hat{m}}(t)}{\sqrt{\pi T}} , \qquad \alpha_0^{\check{m}} = -\sqrt{\frac{T}{\pi}} L^{\check{m}} . \qquad (3.169)$$

3.9 The NG string in light-cone gauge

We shall start with the open string (with free-end b.c.s). We shall impose the gauge conditions

$$X^+(t,\sigma) = x^+(t) , \qquad P_-(t,\sigma) = p_-(t) . \qquad (3.170)$$

It customary to also set $x^+(t) = t$, as for the particle, but it is simpler not to do this. This means that we will not be fixing the gauge completely since we will still be free to make σ-independent reparametrizations of the worldsheet time t.

The above gauge-fixing conditions are equivalent to

$$(P \pm TX')^+ = p_-(t) \quad \Leftrightarrow \quad \alpha_k^+ = 0 \quad \forall k \neq 0 . \qquad (3.171)$$

In other words, we impose a light-cone gauge condition only on the oscillator variables of the string, not on the zero modes (centre of mass variables). Let's check that the gauge has been otherwise fixed. We can investigate this using the criterion summarised by the formula (2.51); we compute

$$\begin{aligned} \left\{ L_n, \alpha_{-k}^+ \right\} &= -ik\alpha_{n-k}^+ \\ &= -ik\alpha_0^+ \delta_{nk} \quad \text{(using gauge condition)} \\ &= -ik\frac{p_-}{\sqrt{\pi T}} \delta_{nk} . \end{aligned} \qquad (3.172)$$

This is invertible if we exclude $n = 0$ and $k = 0$, so we have fixed all but the gauge transformation generated by L_0. Now we have, since $\alpha_k^+ = 0$,

$$\sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} = \sum_{k=1}^{\infty} \frac{i}{k} \dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} , \qquad (3.173)$$

where the $(D-2)$-vectors $\boldsymbol{\alpha}_k$ are the *transverse* oscillator variables.

We also have,

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k = \frac{1}{2\pi T}\left(p^2 + 2\pi T N\right) , \tag{3.174}$$

where N is the *level number*:

$$N = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k . \tag{3.175}$$

For $n \neq 0$,

$$\begin{aligned}
L_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\alpha_k^+ \alpha_{n-k}^- + \alpha_k^- \alpha_{n-k}^+\right) + \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \\
&= \alpha_0^+ \alpha_n^- + \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \qquad \text{(using gauge condition).}
\end{aligned} \tag{3.176}$$

We can solve this for α_n^-; using $p = \sqrt{\pi T}\,\alpha_0$, we get

$$\alpha_n^- = -\frac{\sqrt{\pi T}}{2p_-}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \qquad (n \neq 0). \tag{3.177}$$

As we have solved the constraints $L_n = 0$ for $n \neq 0$, only the $L_0 = 0$ constraint will be imposed by a Lagrange multiplier in the gauge-fixed action, which is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k}\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - \frac{1}{2}e_0\left(p^2 + M^2\right)\right\} , \tag{3.178}$$

where $e_0 = \lambda_0/(\pi T)$ and

$$M^2 = 2\pi T N = N/\alpha' \qquad \left(\alpha' \equiv \frac{1}{2\pi T}\right). \tag{3.179}$$

Notice that the action does not involve α_n^- (for $n \neq 0$) but the Lorentz charges do. Recall that the spin part of the Lorentz charge $\mathcal{J}^{mn}$ is $S^{mn} = -2\sum_{k=1}^{\infty}\frac{i}{k}\alpha_{-k}^{[m}\alpha_k^{n]}$. Its non-zero components of S^{mn} in light-cone gauge are ($I, J, = 1, \ldots D-2$)

$$S^{IJ} = -2\sum_{k=1}^{\infty}\frac{i}{k}\alpha_{-k}^{[I}\alpha_k^{J]} ,$$

$$S^{-I} = -\sum_{k=1}^{\infty}\frac{i}{k}\left(\alpha_{-k}^- \alpha_k^I - \alpha_{-k}^I \alpha_k^-\right) . \tag{3.180}$$

The canonical PB relations that we read off from the action (3.178) are

$$\{x^m, p_n\}_{PB} = \delta_n^m , \qquad \{\alpha_k^I, \alpha_{-k}^J\}_{PB} = -ik\,\delta^{IJ} . \tag{3.181}$$

These may be used to compute the PBs of the Lorentz generators; since $\mathcal{J} = L + S$ where $\{L, S\}_{PB} = 0$, the PB relations among the components of S alone must be the same as those of $\mathcal{J}$. The PBs of S^{IJ} are those of the Lie algebra of the transverse rotation group $SO(D-2)$, and their PBs with S^{-K} are those expected from the fact that S^{-K} is a $(D-2)$ vector. Finally, Lorentz invariance requires that

$$\left\{S^{-I}, S^{-J}\right\}_{PB} = 0 . \tag{3.182}$$

This can be confirmed by making use of the PB relations

$$\left\{\alpha_k^-, \alpha_\ell^-\right\}_{PB} = i\frac{\sqrt{\pi T}}{p_-}(k-\ell)\,\alpha_{k+j}^- , \qquad \left\{\alpha_k^-, \alpha_\ell^I\right\}_{PB} = -i\frac{\sqrt{\pi T}}{p_-}\ell\,\alpha_{k+\ell}^I . \tag{3.183}$$

This has to work because gauge fixing cannot break symmetries; it can only obscure them.

3.9.1 Light-cone gauge for parallel p-plane boundary conditions

What changes if we change the boundary conditions to the mixed Neumann/Dirichlet case of subsubsection 3.8.2? As long as $p > 0$ (so that we have Neumann boundary conditions in at least one space direction, which we use to define the $\pm$ directions) we can still impose the gauge-fixing condition

$$\alpha_k^+ = 0 \quad \forall k \neq 0 , \tag{3.184}$$

and then proceed as before. We can again solve the constraints $L_n = 0$ $(n \neq 0)$ for α_k^- $(k \neq 0)$. The $L_0 = 0$ constraint, which still has to be imposed via a Lagrange multiplier, is

$$0 = \frac{1}{2}\alpha_0^2 + N = \frac{1}{2\pi T}\left(\hat{p}^2 + (TL)^2 + 2\pi TN\right) , \tag{3.185}$$

where

$$\hat{p}^2 = p^{\hat{m}}p_{\hat{m}} , \qquad L = \left|L^{\breve{m}}\right| . \tag{3.186}$$

In other words, the boundary conditions affect only the zero modes. The action is

$$I = \int dt \left\{ \dot{x}^{\hat{m}}p_{\hat{m}} + \sum_{k=1}^{\infty}\frac{i}{k}\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - \frac{1}{2}e_0\left(\hat{p}^2 + M^2\right)\right\} , \tag{3.187}$$

where

$$M^2 = (TL)^2 + N/\alpha' . \tag{3.188}$$

Classically, $N \geq 0$ and the minimum energy configuration has $N = 0$. In this case, $M = TL$, which can be interpreted as the statement that the minimal energy string is a straight string stretched orthogonally between the two p-planes; since they are separated by a distance L the potential energy in the string is TL.

3.9.2 Closed string in light-cone gauge

Now we fix the gauge invariances associated with L_n and $\tilde{L}_n$ for $n \neq 0$ by setting

$$\alpha_k^+ = 0 \quad \& \quad \tilde{\alpha}_k^+ = 0 \qquad \forall k \neq 0 \,. \tag{3.189}$$

This leaves unfixed the gauge invariances generated by L_0 and $\tilde{L}_0$, which are now

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k = \frac{p^2}{8\pi T} + N \,,$$

$$\tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{k=1}^{\infty} \tilde{\boldsymbol{\alpha}}_{-k} \cdot \tilde{\boldsymbol{\alpha}}_k = \frac{p^2}{8\pi T} + \tilde{N} \,. \tag{3.190}$$

Here we have used the closed string relation (3.124) between p and $\alpha_0 = \tilde{\alpha}_0$. By adding and subtracting the two constraints $L_0 = 0$ and $\tilde{L}_0 = 0$ we get the two equivalent constraints

$$p^2 + 4\pi T \left(N + \tilde{N} \right) = 0 \quad \& \quad \tilde{N} - N = 0 \,, \tag{3.191}$$

which will be imposed by the Lagrange multipliers $e_0 = 2(\lambda_0 + \tilde{\lambda}_0)$ and $u_0 = 4\pi T(\lambda - \tilde{\lambda}_0)$ in the gauge-fixed action. The remaining constraints $L_n = 0$ and $\tilde{L}_n = 0$ for $n \neq 0$ we solve for α_k^- and $\tilde{\alpha}_k^-$, as for the open string. The closed string action in light-cone gauge is therefore

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \left(\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} + \dot{\tilde{\boldsymbol{\alpha}}}_k \cdot \tilde{\boldsymbol{\alpha}}_{-k} \right) - \frac{1}{2}e_0 \left(p^2 + M^2 \right) - u_0 \left(N - \tilde{N} \right) \right\} \,, \tag{3.192}$$

where

$$M^2 = 4\pi T \left(N + \tilde{N} \right)$$

$$= 8\pi T N \qquad \left(\text{using } \tilde{N} = N \text{ constraint} \right) \,. \tag{3.193}$$

We now have two constraints, which are first class[10].

4. Interlude: Light-cone gauge in field theory

We shall consider Maxwell's equation, for the vector potential A_m, and the linearised Einstein equations for a symmetric tensor potential h_{mn}, which may be interpreted as the perturbation of the space-time metric about a Minkowski vacuum metric.

We choose light-cone coordinates (x^+, x^-, x^I) $(I = 1, \ldots, D - 2)$. Recall that for the particle we assumed that $p_- \neq 0$, which is equivalent to the assumption that the differential operator ∂_- is invertible. We shall make the same assumption in the application to field theory.

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[10]The constraint function $(N - \tilde{N})$ generates the residual gauge transformation $\delta_\beta \alpha_k(t) = ik\beta(t)\alpha_k(t)$, which originates from invariance under t-dependent shifts of σ.

4.0.3 Maxwell in light-cone gauge

Maxwell's equations are

$$\Box_D A_m - \partial_m (\partial \cdot A) = 0 \,, \qquad m = 0, 1, \ldots, D - 1 \,. \tag{4.1}$$

They are invariant under the gauge transformation $A_m \to A_m + \partial_m \alpha$. The light-cone gauge is

$$A_- = 0 \,. \tag{4.2}$$

To see that the gauge is fixed we set to zero a gauge variation of the gauge-fixing condition: $0 = \delta_\alpha A_- = \partial_- \alpha$. This implies that $\alpha = 0$ if the differential operator ∂_- is invertible.

In this gauge the $m = -$ Maxwell equation is $\partial_- (\partial \cdot A) = 0$, which implies that $\partial \cdot A = 0$ since we assume invertibility of ∂_-. And since $0 = \partial \cdot A = \partial_- A_+ + \partial_I A_I$, we can solve for A_+:

$$A_+ = -\partial_-^{-1} (\partial_I A_I) \,. \tag{4.3}$$

This leaves A_I as the only independent variables. The $m = +$ equation is $\Box_D A_+ = 0$, but this is a consequence of the $m = I$ equation, which is

$$\Box_D A_I = 0 \,, \qquad I = 1, \ldots, D - 2 \,. \tag{4.4}$$

So this is what Maxwell's equations look like in light-cone gauge: wave equations for $D - 2$ independent polarisations.

4.0.4 Linearized Einstein in light-cone gauge

The linearised Einstein equations are

$$\Box_D h_{mn} - 2\partial_{(m} h_{n)} + \partial_m \partial_n h = 0 \,, \qquad h_m \equiv \partial^n h_{nm} \,, \quad h = \eta^{mn} h_{mn} \,. \tag{4.5}$$

They are invariant under the gauge transformation (<u>Exercise</u>: verify this)

$$h_{mn} \to h_{mn} + 2\partial_{(m} \xi_{n)} \,. \tag{4.6}$$

The light-cone gauge choice is

$$h_{-n} = 0 \qquad (n = -, +, I) \qquad \Rightarrow \qquad h_- = 0 \qquad \& \quad h = h_{JJ} \,. \tag{4.7}$$

In the light-cone gauge the "$m = -$" equation is $-\partial_- h_n + \partial_- \partial_n h_{JJ} = 0$, which can be solved for h_n:

$$h_n = \partial_n h_{JJ} \,. \tag{4.8}$$

But since we already know that $h_- = 0$, this tells us that $h_{JJ} = 0$, and hence that $h_n = 0$ and $h = 0$. At this point we see that the equations reduce to $\Box_D h_{mn} = 0$, but

$$\begin{aligned} h_+ = 0 \quad &\Rightarrow \quad h_{++} = -\partial_-^{-1} (\partial_I h_{I+}) \,, \\ h_I = 0 \quad &\Rightarrow \quad h_{+I} = -\partial_-^{-1} (\partial_J h_{JI}) \,, \end{aligned} \tag{4.9}$$

so the only independent components of h_{mn} are h_{IJ}, and this has zero trace. We conclude that the linearised Einstein equations in light-cone gauge are

$$\Box_D h_{IJ} = 0 \quad \& \quad h_{II} = 0 \,. \tag{4.10}$$

The number of polarisation states of the graviton in D dimensions is therefore

$$\frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2}D(D-3) \,. \tag{4.11}$$

For example, for $D = 4$ there are two polarisation states, and the graviton is a massless particle of spin-2.

5. Quantum NG string

Now we pass to the quantum theory. This is simplest in light-cone gauge because this gauge choice removes all unphysical components of the oscillator variables prior to quantization, but it obscures Lorentz invariance. Then we consider how the same results might be found in the conformal gauge, where Lorentz invariance is still manifest.

5.1 Light-cone gauge quantization: open string

The canonical PB relations of the open string in Fourier modes are (3.181). These become the canonical commutation relations

$$[\hat{x}^m, \hat{p}_n] = i\delta_n^m \,, \qquad \left[\hat{\alpha}_k^I, \hat{\alpha}_{-k}^J\right] = k\,\delta^{IJ} \,, \tag{5.1}$$

where the hats now indicate operators, and the hermiticity of the operators $(\hat{X}, \hat{P})$ requires that

$$\hat{\boldsymbol{\alpha}}_{-k} = \hat{\boldsymbol{\alpha}}_k^\dagger \,. \tag{5.2}$$

A state of the string of definite momentum is the tensor product of a momentum eigenstate $|p\rangle$ with a state in the oscillator Fock space, built upon the Fock vacuum state $|0\rangle$ annihilated by all annihilation operators:

$$\hat{\boldsymbol{\alpha}}_k|0\rangle = \mathbf{0} \quad \forall\, k \in \mathbb{Z}^+ \,. \tag{5.3}$$

We get other states in the Fock space by acting on the oscillator vacuum with the creation operators $\hat{\boldsymbol{\alpha}}_{-k}$ any number of times, and for any $k > 0$. This gives us a basis for the entire infinite-dimensional space.

Next, we need to replace the level number N by a level number operator $\hat{N}$, but there is an operator ordering ambiguity; different orderings lead to operators $\hat{N}$ that differ by a constant. We shall choose to call $\hat{N}$ the particular operator that annihilates the oscillator vacuum; i.e.

$$\hat{N} = \sum_{k=1}^{\infty} \hat{\boldsymbol{\alpha}}_{-k} \cdot \hat{\boldsymbol{\alpha}}_k \qquad \Rightarrow \qquad \hat{N}|0\rangle = 0 \,. \tag{5.4}$$

So the oscillator vacuum has level number zero. This removes the ambiguity in the definition of $\hat{N}$ but it does *not* remove the ambiguity in passing from the classical to the quantum theory; whenever we see N in the classical theory we may still replace it by $\hat{N}$ plus a constant in the quantum theory.

Notice now that

$$\left[\hat{N}, \hat{\boldsymbol{\alpha}}_{-k}\right] = k\,\hat{\boldsymbol{\alpha}}_{-k}\,. \tag{5.5}$$

This tell us that acting on a state with any component of $\hat{\boldsymbol{\alpha}}_{-k}$ raises the level number by k, and this tells that $\hat{N}$ is diagonal in the Fock state basis constructed in the way described above, and that the possible level numbers (eigenvalues of $\hat{N}$) are $N = 0, 1, 2, \ldots, \infty$. We can therefore organise the states according to their level number. There is only one state in the Fock space with $N = 0$, the oscillator vacuum. At $N = 1$ we have the $(D - 2)$ states $\hat{\alpha}^I_{-1}|0\rangle$. At $N = 2$ we have the states

$$\hat{\alpha}^I_{-2}|0\rangle\,, \qquad \hat{\alpha}^I_{-1}\hat{\alpha}^J_{-1}|0\rangle\,. \tag{5.6}$$

At $N = 3$ we have the states

$$\hat{\alpha}^I_{-3}|0\rangle\,, \qquad \hat{\alpha}^I_{-2}\hat{\alpha}^J_{-1}|0\rangle\,, \qquad \hat{\alpha}^I_{-1}\hat{\alpha}^J_{-1}\hat{\alpha}^K_{-1}|0\rangle\,, \tag{5.7}$$

and so on.

A generic state of the string at level N in a momentum eigenstate takes the form

$$|p\rangle \otimes |\Psi_N\rangle\,, \tag{5.8}$$

where p is the D-momentum and Ψ_N some state in the oscillator Fock space with level number N. The mass-shell constraint for such a state implies that $p^2 = -M^2$, where

$$M^2 = 2\pi T\left(N - a\right)\,. \tag{5.9}$$

The constant a is introduced to take care of the operator ordering ambiguity in passing from the classical to the quantum theory.

If we had defined $\hat{N}$ using the conventional Weyl ordering that leads to the usual zero-point energy for a harmonic oscillator, we would find that its eigenvalues are not N but rather

$$N + \frac{(D - 2)}{2}\sum_{k=1}^{\infty} k\,. \tag{5.10}$$

This is because the $(D - 2)$ oscillators associated to the pair $(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}^\dagger_k)$ have angular frequency $|k|$, and we have to sum over all oscillators. This would lead us to make the identification

$$-a = \frac{(D - 2)}{2}\sum_{k=1}^{\infty} k\,. \tag{5.11}$$

The sum on the RHS is infinite, it would seem. In fact, it is ill-defined. One way to define it is as the $s \to -1$ limit of

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} . \tag{5.12}$$

When this sum converges, it defines a function that can be analytically continued to the entire complex s-plane except at $s = 1$, where it has a simple pole; this is the Riemann zeta function and

$$\zeta(-1) = -\frac{1}{12} . \tag{5.13}$$

Using this in (5.11) we find that

$$a = \frac{(D-2)}{24} . \tag{5.14}$$

This looks rather dubious, so let's leave it aside for the moment and proceed to analyse the string spectrum for arbitrary a, and level by level. We shall use the standard notation

$$2\pi T = 1/\alpha' . \tag{5.15}$$

- $N = 0$. There is one state, and hence a scalar, with $M^2 = -a/\alpha'$. For $a > 0$ (as (5.14) suggests) this scalar is a tachyon.

- $N = 1$. There are now $(D-2)$ states, $\hat{\alpha}_{-1}|0\rangle$ with $M^2 = (1-a)/\alpha'$. The only way that these states could be part of a Lorentz-invariant theory is if they describe the polarization states of a *massless* vector (a massive vector has $(D-1)$ polarisation states), so Lorentz invariance requires

$$a = 1 . \tag{5.16}$$

- $N = 2$. Since $a = 1$ the $N = 2$ states are massive, with $M^2 = 1/\alpha'$. The states are those of (5.6), which are in the symmetric 2nd-rank tensor plus vector representation of $SO(D-2)$. These states form a symmetric traceless tensor of $SO(D-1)$ and hence describe a massive spin-2 particle[11].

We now know that the string ground state is a scalar tachyon, and its first excited state is a massless vector, a "photon". All higher level states are massive, and so should be in $SO(D-2)$ representations that can be combined to form $SO(D-1)$ representations (i.e. representations of the rotation group). We have seen that this is true for $N = 2$ and it can be shown to be true for all $N \geq 2$. The $N = 1$ states are exceptional in this respect.

Notice that if $a = 1$ is used in (5.14) we find that $D = 26$. Remarkably, it is indeed true that Lorentz invariance requires $D = 26$.

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[11]A symmetric traceless tensor field of rank n is usually said to describe a particle of "spin n" even though "spin" is not sufficient to label states in space times of dimension $D > 4$.

5.1.1 Critical dimension

Because Lorentz invariance is not *manifest* in light-cone gauge it might be broken when we pass to the quantum theory. To check Lorentz invariance we have to compute the commutators of the quantum Lorentz charges $\hat{\mathcal{J}}^{mn}$. In fact, $\hat{\mathcal{J}} = \hat{L} + \hat{S}$ and it is easy to see that $[\hat{L}, \hat{S}] = 0$ so we can focus on the spin $\hat{S}^{mn}$; its components should obey the same algebra as those of $\hat{\mathcal{J}}$, and this requires that

$$\left[\hat{S}^{-I}, \hat{S}^{-J} \right] = 0 . \tag{5.17}$$

If the $\{,\}_{PB} \to -i[,]$ rule were to apply to these charges then Lorentz invariance of the quantum string would be guaranteed because the classical theory is Lorentz invariant, even in the light-cone gauge. But it does not apply because the S^{-I} are cubic in the transverse oscillators; a product of two of them is therefore 6th-order in transverse oscillators, but the commutator reduces this to 4th order. The classical PB computation gives zero for this 4th order term, but to achieve this in the quantum theory we might have to change the order of operators, which would produce a term quadratic in oscillators. So, potentially, the RHS of (5.17) might end up being an expression quadratic in transverse oscillators.

Because of this possibility, we need to check (5.17); there is no guarantee that it will be true. We can do this calculation once we have the quantum analogs of the PB relations (3.183). The commutator $[\hat{\alpha}_k^-, \hat{\alpha}_\ell^-]$, for $k\ell \neq 0$ is the one we have to examine carefully[12]. There is no ordering ambiguity in the quantum version of the expression (3.177) for α_n^-, so we are taking the commutator of well-defined operators. Looking first at the $k + \ell = 0$ case, we find that[13]

$$\left[\hat{\alpha}_k^-, \hat{\alpha}_{-k}^- \right] = 2k \frac{\pi T}{p_-^2} \left(\frac{|\mathbf{p}|^2}{2\pi T} + \hat{N} \right) + \frac{2\pi T}{p_-^2} \left(\frac{D-2}{24} \right) \left(k^3 - k \right) . \tag{5.18}$$

Using the mass-shell condition in the operator form

$$\left[\frac{|\mathbf{p}|^2}{2\pi T} + \hat{N} \right] = \left[a - 2 \frac{p_+ p_-}{2\pi T} \right] , \tag{5.19}$$

which is valid when the operators act on any physical state, and using the fact that $p_+ = \sqrt{\pi T} \, \alpha_0^-$, we can rewrite (5.18) as

$$\left[\hat{\alpha}_k^-, \hat{\alpha}_{-k}^- \right] = -2k \frac{\sqrt{\pi T}}{p_-} \alpha_0^- + \frac{2\pi T}{p_-^2} \left(k \left[a - \frac{(D-2)}{24} \right] + \frac{(D-2))}{24} k^3 \right) \tag{5.20}$$

More generally, one finds that

$$\left[\hat{\alpha}_k^-, \hat{\alpha}_\ell^- \right] = -\frac{\sqrt{\pi T}}{p_-} \left(k - \ell \right) \hat{\alpha}_{k+\ell}^- + \frac{2\pi T}{p_-^2} \left(k \left[a - \frac{(D-2)}{24} \right] + \frac{(D-2)}{24} k^3 \right) \delta_{k+\ell} \tag{5.21}$$

---

[12]Recall that $\alpha_0^- \propto p_+$, which is still an independent variable in the version of the light-cone gauge used here.

[13]A very similar calculation will be explained in detail later.

where

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \,. \end{cases} \tag{5.22}$$

Compare this result with the analogous PB relation of (3.183); the second term in (5.21) has no classical counterpart. Using this result leads to the further result that

$$\left[\hat{S}^{-I}, \hat{S}^{-J}\right] = \frac{4\pi T}{p_-^2} \sum_{k=1}^{\infty} \left(\left[\frac{(D-2)}{12} - 2 \right] k + \frac{1}{k} \left[2a - \frac{(D-2)}{12} \right] \right) \hat{\alpha}_{-k}^{[I} \hat{\alpha}_k^{J]} \,, \tag{5.23}$$

which is zero for $D > 3$ iff

$$a = 1 \quad \& \quad D = 26 \,. \tag{5.24}$$

We therefore confirm that Lorentz invariance requires $a = 1$, but we now see that it also requires $D = 26$; this is the *critical dimension* of the NG string.

From now on we drop the hats on operators.

5.1.2 Quantum string with ends on a p-plane: Dirichlet branes

We shall consider only the case for which both ends of an open string are constrained to move in the *same p*-plane (so $L = 0$), and we shall assume that $p \geq 3$. In this case the boundary conditions preserve an $SO(D - p - 1)$ subgroup of the $SO(D - 2)$ transverse rotation group, so we write

$$I = (\hat{I}, \check{I}) \qquad \hat{I} = 1, \ldots p - 1 \,, \quad \check{I} = 1, \ldots, D - p - 1 \quad \Rightarrow \quad \hat{m} = (+, -, \hat{I}) \,. \tag{5.25}$$

This means, for example, that (the hats here are **not** "operator hats")

$$N = \sum_{k=1}^{\infty} \left(\hat{\alpha}_{-k} \cdot \hat{\alpha}_k + \check{\alpha}_{-k} \cdot \check{\alpha}_k \right) \,. \tag{5.26}$$

Quantization proceeds exactly as for the string with free ends, *except that the mass-shell condition at given level N is now a wave-equation in the $(p+1)$-dimensional Minkowski space-time.* The mass-squared at level N is again $2\pi T(N - a)$, and the $N = 1$ excited states are

$$|\hat{p}\rangle \otimes \left[A_{\hat{I}}(\hat{p}) \alpha_{-1}^{\hat{I}} + A_{\check{I}}(\hat{p}) \alpha_{-1}^{\check{I}} \right] |0\rangle \,. \tag{5.27}$$

We can identify $A_{\hat{I}}$ as the $(p-2)$ physical components of a $(p+1)$-vector potential, and $A_{\check{I}}$ as $(D-p-1)$ scalars, all propagating in the Mink_{p+1} subspace of Mink_D. Because a massive photon would have $(p - 1)$ physical components, it must be massless (we can't use one of the $(D-p-1)$ scalars as the extra component of the massive photon field because this would break the $SO(D - p - 1)$ transverse rotation group). Again, this tells us that $a = 1$. To verify that the $(p + 1)$-dimensional Lorentz invariance is preserved in the quantum theory we need to check that

$$\left[S^{-\hat{I}}, S^{-\hat{J}} \right] = 0 \,, \tag{5.28}$$

and this again turns out to be true only if $a = 1$ and $D = 26$.

Because the b.c.s break translation invariance in the directions orthogonal to the fixed p-plane on which the string ends, the string can lose or gain momentum in these directions. This was considered unphysical for many years, but there is a simple physical interpretation. The $(D - p - 1)$ massless scalars propagating on the p-plane represent fluctuations of this p-plane in the $(D - p - 1)$ space directions orthogonal to the plane. This is exactly what one would expect of a dynamical p-brane. This is one way in which "branes" appear in string theory, in this case as boundaries on which open strings may end; these are known as Dirichlet branes, or "D-branes" (or "Dp-brane" for a p-brane). However, the ground state is still a tachyon, which indicates that NG D-branes are unstable.

5.1.3 Quantum closed string

Finally, we consider light-cone gauge quantization of the closed string. There are now two sets of oscillator operators, with commutation relations

$$\left[\alpha_k^I, \alpha_{-k}^J\right] = k\delta^{IJ} = \left[\tilde{\alpha}_k^I, \tilde{\alpha}_{-k}^J\right] . \tag{5.29}$$

The oscillator vacuum is now

$$|0\rangle = |0\rangle_R \otimes |0\rangle_L , \qquad \boldsymbol{\alpha}_k |0\rangle_R = 0 \quad \& \quad \tilde{\boldsymbol{\alpha}}_k |0\rangle_L = 0 \qquad \forall k > 0 . \tag{5.30}$$

We define the level number operators N and $\tilde{N}$ (*hats dropped*) such that they annihilate the oscillator vacuum. Again, we can choose a basis for the Fock space built on the oscillator vacuum for which these operators are diagonal, with eigenvalues N and $\tilde{N}$. In the space of states of definite p and definite $(N, \tilde{N})$ the mass-shell and level matching constraints are

$$p^2 + 4\pi T \left[(N - a_R) + \left(\tilde{N} - a_L\right)\right] = 0 \quad \& \quad (N - a_R) = \left(\tilde{N} - a_L\right) . \tag{5.31}$$

Since we want $|0\rangle$ to be a physical state we must choose $a_L = a_R = a$, and then we have

$$p^2 + 8\pi T (N - a) = 0 \quad \& \quad \tilde{N} = N . \tag{5.32}$$

This means that we can organise the states according to the level N, with $M^2 = 8\pi T(N - a)$. We must do this respecting the level-matching condition $\tilde{N} = N$. Let's look at the first few levels

- $N = 0$. There is one state, and hence a scalar, with $M^2 = -4a/\alpha'$.

- $N = 1$. There are now $(D - 2) \times (D - 2)$ states

$$\alpha_{-1}^I |0\rangle_R \otimes \tilde{\alpha}_{-1}^J |0\rangle_L \tag{5.33}$$

We can split these into irreducible representations by taking the combinations

$$[h_{IJ}(p) + \delta_{IJ}\phi(p) + b_{IJ}(p)] \, \alpha^I_{-1}|0\rangle_R \otimes \tilde{\alpha}^J_{-1}|0\rangle_L \,, \qquad (5.34)$$

where h_{IJ} is symmetric traceless tensor, b_{IJ} an antisymmetric tensor and ϕ a scalar. The only way that these could be part of a Lorentz-invariant theory is if h_{IJ} are the physical components of a *massless* spin-2 field because massive spin-2 would require a symmetric traceless tensor of the full rotation group $SO(D-1)$. Then b_{IJ} must be the physical components of a massless antisymmetric tensor field, and ϕ a massless scalar (the dilaton).

Since we require $M^2 = 0$ we must choose $a = 1$ again[14]. This means that the ground state is a tachyon, indicating an instability of the Minkowski vacuum.

- $N = 2$. Since $a = 1$, the $N = 2$ states are massive, with $M^2 = 8\pi T$. Recalling that the open string states at level 2 combined into a symmetric traceless tensor of $SO(D-1)$, we see that the level-2 states of the closed string will combine into those $SO(D-1)$ representations found in the product of two symmetric traceless $SO(D-1)$ tensors. This includes a 4th-order totally symmetric traceless tensor describing a particle of spin-4; there will be several lower spins too.

The most remarkable fact about these results is that the closed string spectrum contains a massless spin-2 particle, suggesting that a closed string theory will be a theory of quantum gravity. As for the open string, one finds that Lorentz invariance is preserved only if $D = 26$ (the calculation needed to prove this is a repeat of the open string calculation because the spin operator is a sum of a contribution from "left" oscillators and a contribution from "right" oscillators). The ground state is a tachyon, but the tachyon is absent in superstring theory, for which the critical dimension is $D = 10$, and there are various ways to compactify dimensions so as to arrive at more-or-less realistic models of gravity coupled to matter.

5.2 "Old covariant" quantization

Dirac's method of dealing with first-class constraints would appear to allow us to quantise the string in a way that preserves manifest Lorentz invariance. Let's consider the open string with free-end b.c.s. Recall that the action in terms of Fourier modes is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n \in \mathbb{Z}} \lambda_{-n} L_n \right\} \,, \qquad L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \cdot \alpha_{n-k} \,, \quad (5.35)$$

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[14]This is usually given as $a = 2$ but that's due to a different definition of a for the closed string.

and that $\{L_k, L_\ell\}_{PB} = -i\,(k - \ell)\,L_{k+\ell}$. Applying the $\{,\} \to -i[,]$ rule to the PBs of the canonical variables, we get the canonical commutation relations

$$[x^m, p_n] = i\delta^m_n\,, \qquad [\alpha^m_k, \alpha^n_{-k}] = k\eta^{mn}\,. \tag{5.36}$$

Now we define the oscillator vacuum $|0\rangle$ by

$$\alpha^m_k|0\rangle = 0 \quad \forall k > 0 \qquad (m = 0, 1, \ldots, D - 1). \tag{5.37}$$

The Fock space is built on $|0\rangle$ by the action of the creation operators α^m_{-k}, but this gives a space with many unphysical states since we now have D creation operators for each k, whereas we know (from light-cone gauge quantization) that $D - 2$ suffice to construct the physical states.

Can we remove unphysical states by imposing the physical state conditions

$$L_n|\text{phys}\rangle = 0 \quad \forall n \text{ ?} \tag{5.38}$$

If this were possible then we would have achieved a Lorentz covariant quantization of the massless spin-1 particle at level 1 without the need for unphysical polarisation states, but this is not possible: the Lorentz-invariant Lorenz gauge $p \cdot A = 0$ reduces their number from D to $D - 1$ but no Lorentz-invariant condition will reduce it to $D - 2$. So we are going to run into a problem!

Notice that we do not encounter an operator ordering ambiguity when passing from the classical phase-space function L_n to the corresponding operator L_n except when $n = 0$, so the operator L_n is unambiguous for $n \neq 0$ and it is easy to see that

$$L_n|0\rangle = 0\,, \qquad n > 0\,. \tag{5.39}$$

However, it is also easy to see [exercise: check these statements] that

$$L_{-1}|0\rangle \equiv \frac{1}{2}\sum_k \alpha_k \cdot \alpha_{-1-k}|0\rangle = \alpha_0 \cdot \alpha_{-1}|0\rangle$$

$$L_{-2}|0\rangle \equiv \frac{1}{2}\sum_k \alpha_k \cdot \alpha_{-2-k}|0\rangle = \left(\alpha_0 \cdot \alpha_{-2} + \frac{1}{2}\alpha^2_{-1}\right)|0\rangle\,, \tag{5.40}$$

so it looks as though not even $|0\rangle$ is physical. In fact, there are no states in the Fock space satisfying (5.38) because the algebra of the operators L_n has a quantum anomaly, which is such that the set of operators $\{L_n; n \in \mathbb{Z}\}$ is not "first-class". That is what we shall now prove.

Since the L_n are quadratic in oscillator variables, the product of two of them is quartic but the commutator $[L_m, L_n]$ is again quadratic. That is what we expect from the PB, which is proportional to L_{m+n}, but to get the operator L_{m+n} from the expression that results from computing the commutator $[L_m, L_n]$, we may need to re-order operators, and that would produce a constant term. So, we must find that

$$[L_m, L_n] = (m - n)\,L_{m+n} + A_{mn} \tag{5.41}$$

for some constants A_{mn}. We can compute the commutator using the fact that

$$[L_m, \alpha_k] = -k\,\alpha_{k+m}\,. \qquad (5.42)$$

This can be verified directly but it also follows from the corresponding PB result because no ordering ambiguity is possible either on the LHS or the RHS. Using this, we find that

$$[L_m, L_n] = \sum_k \left([L_m, \alpha_k] \cdot \alpha_{n-k} + \alpha_k \cdot [L_m, \alpha_{n-k}]\right)$$
$$= -\frac{1}{2}\sum_k k\,\alpha_{k+m} \cdot \alpha_{n-k} - \frac{1}{2}\sum_k (n-k)\,\alpha_k \cdot \alpha_{n+m-k}\,. \qquad (5.43)$$

As long as $n + m \neq 0$ this expression is not affected by any change in the order of operators, so it must equal what one gets from an application of the $\{,\}_{PB} \to -i[,]$ rule. In other words, $A_{mn} = 0$ unless $m + n = 0$. We can check this by using the fact that $\alpha_{-k} = \alpha_k^\dagger$, so that

$$\alpha_k|0\rangle = 0 \quad \Leftrightarrow \quad \langle 0|\alpha_{-k} = 0\,. \qquad (5.44)$$

From this we see that for $m + n \neq 0$,

$$\langle 0|\alpha_{k+m} \cdot \alpha_{n-k}|0\rangle = 0 = \langle 0|\alpha_k \cdot \alpha_{n+m-k}|0\rangle \qquad (m+n \neq 0)\,, \qquad (5.45)$$

and hence that $\langle 0|[L_m, L_n]|0\rangle = 0$ unless $m + n = 0$. This tells us that $A_{mn} = A(m)\delta_{m+n}$.

We now focus on the $m + n = 0$ case, for which

$$[L_m, L_{-m}] = 2mL_0 + A(m) \quad \Rightarrow \quad A(-m) = -A(m)\,. \qquad (5.46)$$

Because of an operator ordering ambiguity, the operator L_0 is only defined up to the addition of a constant, so what we find for $A(m)$ will obviously depend on how we define the operator L_0. We shall define it as

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k\,, \qquad (5.47)$$

but we should keep in mind that it is possible to redefine L_0 by adding a constant to it. We could now return to (5.43), set $n = -m$, and then complete the computation to find $A(m)$. This can be done, but it has to be done with great care to avoid illegitimate manipulations of infinite sums. For that reason we here take an indirect route.

First we use the Jacobi identity[15]

$$[L_k, [L_m, L_n]] + \text{cyclic permutations} \equiv 0\,, \qquad (5.48)$$

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[15]This is a consequence of the associativity of the product of operators; i.e. we use the fact that $(L_k L_m)L_n = L_k(L_m L_n)$. The antisymmetric Lie product of two operators defined by their commutator is *not* associative (the Jacobi identity tells us that) but what is relevant here is the product used to define the commutator, not the Lie product defined by the commutator.

to deduce that [Exercise]

$$[(m - n)A(k) + (n - k)A(m) + (k - m)A(n)]\,\delta_{m+n+k} = 0\,. \tag{5.49}$$

Now set $k = 1$ and $m = -n - 1$ (so that $m + n + k = 0$) to deduce that

$$A(n + 1) = \frac{(n + 2)A(n) - (2n + 1)A(1)}{n - 1} \qquad n \geq 2\,. \tag{5.50}$$

This is a recursion relation that determines $A(n)$ for $n \geq 3$ in terms of $A(1)$ and $A(2)$, so there are two independent solutions of the recursion relation. You may verify that $A(m) = m$ and $A(m) = m^3$ are solutions, so now we have

$$[L_m, L_{-m}] = 2mL_0 + c_1 m + c_2 m^3\,, \tag{5.51}$$

for some constants c_1 and c_2. Observing that $(m > 0)$

$$\langle 0|\,[L_m, L_{-m}]\,|0\rangle = \langle 0|L_m L_{-m}|0\rangle = ||L_{-m}|0\rangle||^2\,, \tag{5.52}$$

and that

$$\langle 0|L_0|0\rangle = \frac{1}{2}\alpha_0^2 = \frac{p^2}{2\pi T}\,, \tag{5.53}$$

we deduce that

$$||L_{-m}|0\rangle||^2 - \left(\frac{p^2}{\pi T}\right) m = c_1 m + c_2 m^3\,. \tag{5.54}$$

We can now get two equations for the two unknown constants (c_1, c_2) by evaluating $||L_{-m}|0\rangle||^2$ for $m = 1$ and $m = 2$. Using (5.40) we find that

$$||L_{-1}|0\rangle||^2 = \frac{1}{\pi T}\langle 0|p \cdot \alpha_1\, p \cdot \alpha_{-1}|0\rangle = \frac{p_m p_n}{\pi T}\langle 0|\alpha_1^m \alpha_{-1}^n|0\rangle = \frac{p^2}{\pi T}\,, \tag{5.55}$$

and that

$$\begin{aligned}
||L_{-2}|0\rangle||^2 &= \langle 0|\left(\alpha_0 \cdot \alpha_2 + \frac{1}{2}\alpha_1^2\right)\left(\alpha_0 \cdot \alpha_{-2} + \frac{1}{2}\alpha_{-1}^2\right)|0\rangle \\
&= \frac{1}{\pi T}\langle 0|p \cdot \alpha_2\, p \cdot \alpha_{-2}|0\rangle + \frac{1}{4}\langle 0|\alpha_1^2 \alpha_{-1}^2|0\rangle \\
&= \frac{2p^2}{\pi T} + \frac{D}{2}\,,
\end{aligned} \tag{5.56}$$

from which we see that

$$||L_{-1}|0\rangle||^2 - \frac{p^2}{\pi T} = 0\,, \qquad ||L_{-2}|0\rangle||^2 - \frac{2p^2}{\pi T} = \frac{D}{2}\,, \tag{5.57}$$

and hence that

$$c_1 + c_2 = 0\,, \qquad c_1 + 4c_2 = \frac{D}{4} \qquad \Rightarrow \qquad c_2 = -c_1 = \frac{D}{12}\,. \tag{5.58}$$

Inserting this result into (5.51), we have

$$[L_m, L_{-m}] = 2mL_0 + \frac{D}{12}\left(m^3 - m\right),$$ (5.59)

and hence that

$$[L_m, L_n] = (m - n)\, L_{m+n} + \frac{D}{12}\left(m^3 - m\right)\delta_{m+n}.$$ (5.60)

This is an example of the *Virasoro algebra*. In general, the Virasoro algebra takes the form

$$[L_m, L_n] = (m - n)\, L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n},$$ (5.61)

where c is the *central charge*. In the current context we get this algebra with $c = D$.

5.2.1 The Virasoro constraints

We have just seen that without breaking manifest Lorentz covariance, it is not possible to impose the physical state conditions that we would need to impose to eliminate all unphysical degrees of freedom. In view of our light-cone gauge results, this should not be too much of a surprise. We saw that the level-one states of the open string are the polarisation states of a massless spin-1 particle, but a *manifestly* covariant description of a massless spin-1 particle *necessarily* involves unphysical degrees of freedom.

This argument suggests that we should aim to impose weaker conditions that leave unphysical states associated to gauge invariances, but that remove all other unphysical states. For correspondence with the classical NG string in a semi-classical limit, these weaker conditions should have the property that

$$\langle\Psi'|\left(L_n - a\delta_n\right)|\Psi\rangle = 0 \quad \forall n,$$ (5.62)

for any two allowed states $|\Psi\rangle$ and $|\Psi'\rangle$. Because of the operator ordering ambiguity in L_0, we should allow for the possibility that the L_0 operator of relevance here is shifted by some constant relative to how we defined it in (5.47), hence the constant a (which will turn out to be the same constant a that we introduced in the light-cone gauge quantization). We can achieve (5.62) without encountering inconsistencies by imposing the *Virasoro constraints*

$$L_n|\Psi\rangle = 0 \quad \forall n > 0 \quad \& \quad (L_0 - a)\,|\Psi\rangle = 0.$$ (5.63)

Let's call states $|\Psi\rangle$ satisfying the Virasoro constraints "Virasoro-allowed".

As for the light-cone gauge quantization, we can define the level number operator

$$N = \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k.$$ (5.64)

This differs from the level-number operator of the light-cone gauge in that it includes all D components of the oscillator annihilation and creation operators, not just the $D - 2$ transverse components. Otherwise it plays a similar role. We can choose a basis such that the operators p and N are diagonal, in which case p and N will mean their respective eigenvalues. For given p and N, the Virasoro mass-shell condition is

$$p^2 + M^2 = 0\,, \qquad M^2 = 2\pi T\left(N - a\right)\,. \tag{5.65}$$

The mass-squared is a linearly increasing function of the level number, as in the light-cone case. Let's consider the first few levels. The only $N = 0$ state in the oscillator Fock space is $|0\rangle$ (which gives the string state $|p\rangle \otimes |0\rangle$).

The general $N = 1$ state is

$$A_m(p)\,\alpha_{-1}^m|0\rangle\,, \qquad p^2 = 2\pi T(a - 1)\,. \tag{5.66}$$

The norm-squared of this state is

$$||A \cdot \alpha_{-1}|0\rangle||^2 = A_m A_n \langle 0|\alpha_1^m \alpha_{-1}^n|0\rangle = \eta^{mn} A_m A_n \equiv A^2\,. \tag{5.67}$$

This could be negative but we still have to impose the other Virasoro conditions $L_k(A \cdot \alpha_{-1})|0\rangle = 0$ for $k > 0$. For $k \geq 2$ these conditions are trivially satisfied, but for $k = 1$ we find that

$$0 = L_1(A \cdot \alpha_{-1})|0\rangle = A \cdot \alpha_0|0\rangle \quad \Rightarrow \quad p \cdot A = 0\,. \tag{5.68}$$

Let's now consider the implications of this for the constant a:

- $a > 1$. Then $p^2 > 0$, so p is spacelike. In a frame where $p = (0; p, \mathbf{0})$ the constraint $p \cdot A = 0$ is equivalent to $A_1 = 0$, so the general allowed level-1 state in this frame is

$$|\Psi_1\rangle = A_0\,\alpha_{-1}^0|0\rangle + \mathbf{A} \cdot \boldsymbol{\alpha}_{-1}|0\rangle\,, \qquad |||\Psi_1\rangle||^2 = -A_0^2 + |\mathbf{A}|^2\,. \tag{5.69}$$

  The state with $\mathbf{A} = \mathbf{0}$ has negative norm; it is a "ghost". This implies a violation of unitarity (non-conservation of probability) so we should not allow $a > 1$.

- $a < 1$. Then $p^2 < 0$, so p is timelike. In a frame where $p = (p, \vec{0})$, the constraint $p \cdot A = 0$ implies that $A_0 = 0$, so the general allowed level-1 state in this frame is

$$|\Psi_1\rangle = \vec{A} \cdot \vec{\alpha}_{-1}|0\rangle\,, \qquad |||\Psi_1\rangle||^2 = |\vec{A}|^2\,. \tag{5.70}$$

  Now, all non-zero allowed states have positive norm. There are $(D - 1)$ independent such states, exactly the number required for a massive spin-1 particle.

This does not agree with the light-cone gauge result. Quantization has partially broken the gauge invariance of the classical NG string, and this has resulted in an additional degree of freedom that was absent classically. There is nothing obviously unphysical about this extra degree of freedom but it has no classical analog, so we cannot really say that we have a quantum NG string.

- $a = 1$. Then $p^2 = 0$, which agrees with the light-cone gauge result. In a frame where $p = (1; 1, \mathbf{0})$ the constraint $p \cdot A = 0$ implies that $A_0 = A_1$, and hence that the general allowed level-1 state is

$$|\Psi_1\rangle = \sqrt{2}\, A_1 \cdot \alpha^+_{-1}|0\rangle + \mathbf{A} \cdot \boldsymbol{\alpha}_{-1}|0\rangle \,, \qquad ||\,|\Psi_1\rangle||^2 = |\mathbf{A}|^2 \geq 0 \,. \qquad (5.71)$$

There are no ghosts, but the state $\alpha^+_{-1}|0\rangle$ is orthogonal to all allowed states, *including itself*, which implies that it is a null state:

$$\left|\left|\alpha^+_{-1}|0\rangle\right|\right|^2 = \langle 0|\alpha^+_1 \alpha^+_{-1}|0\rangle = \eta^{++} = 0 \,. \qquad (5.72)$$

Although the number of independent allowed level-1 states is still $(D - 1)$ we may identify any two states that differ by the addition of some multiple of $\alpha^+_{-1}|0\rangle$; in other words we consider the equivalence class of allowed states defined by the equivalence relation

$$|\Psi_1\rangle \sim |\Psi_1\rangle + c\,\alpha^+_{-1}|0\rangle \,, \qquad (5.73)$$

for any complex number c. Because $\alpha^+_{-1}|0\rangle$ is orthogonal to all physical states, it has no effect on the matrix of inner products of allowed states. The dimension of the space of these equivalence classes is $(D-2)$ because the basis state $\alpha^+_{-1}|0\rangle$ is now equivalent to the zero state. If physical states are defined in this way we get agreement with the light-cone gauge.

What we are finding here is essentially the Gupta-Bleuler quantization of electrodynamics (in D space-time dimensions).

Let's now look at the level-2 states (for which $\alpha_0^2 = -2$). The general level-2 state is

$$|\Psi_2\rangle = \left(A_{mn}\alpha^m_{-1}\alpha^n_{-1} + B_m\alpha^m_{-2}\right)|0\rangle \qquad (5.74)$$

This is trivially annihilated by L_k for $k > 2$. However, $L_1|\Psi_2\rangle = 0$ imposes the condition

$$B_n = -\alpha_0^m A_{mn} \,, \qquad (5.75)$$

and $L_2|\Psi_2\rangle = 0$ imposes the condition

$$\eta^{mn} A_{mn} = -2\alpha_0 \cdot B \,. \qquad (5.76)$$

This means that only the traceless part of A_{mn} is algebraically independent, so the dimension of the Virasoro-allowed level-2 space is

$$\frac{1}{2}D(D+1) - 1 = \left[\frac{1}{2}D(D-1) - 1\right] + D \tag{5.77}$$

The dimension is D larger than the physical level-2 space that we found from light-cone gauge quantization (that space was spanned by the polarisation states of a massive spin-2 particle, so a symmetric traceless tensor of the $SO(D-1)$ rotation group). However, equivalence with the light-cone gauge results is still possible if there are sufficient null states, and no ghosts.

To analyse this we need to consider the norm-squared of $|\Psi_2\rangle$, which is

$$||\,|\Psi_2\rangle\,||^2 = 2A^{mn}A_{mn} + 2B^2\,. \tag{5.78}$$

Then we need to consider the implications for this norm of (5.75) and (5.76). We will not carry out a complete analysis; the final result is that there are no ghosts only if $D \leq 26$ and then there are sufficient null states for equivalence with the light-cone gauge results iff $D = 26$.

It is simple to see that there are ghosts if $D > 26$, and null states if $D = 26$. Consider the special case of (5.74) for which

$$A^{mn} = \eta^{mn} + k_1 \alpha_0^m \alpha_0^n\,, \qquad B^m = k_2 \alpha_0^m\,. \tag{5.79}$$

This gives us the *Lorentz scalar* state

$$\left[\alpha_{-1}^2 + k_1 \left(\alpha_0 \cdot \alpha_{-1}\right)^2 + k_2 \alpha_0 \cdot \alpha_{-2}\right]|0\rangle\,. \tag{5.80}$$

The conditions (5.75) and (5.76) determine the constants (k_1, k_2) to be

$$k_1 = \frac{D+4}{10}\,, \qquad k_2 = \frac{D-1}{5}\,, \tag{5.81}$$

and then one finds that the norm-squared is

$$-\frac{2}{25}\,(D-1)\,(D-26)\,. \tag{5.82}$$

This is negative for $D > 26$, so the state being considered is a ghost. To avoid ghosts we require $D \leq 26$.

For $D < 26$ the scalar state (5.80) has positive norm, and this implies that there is a physical scalar field at level 2, in disagreement with the result of light-cone gauge quantization. If fact, for $D < 26$ one finds not only an additional scalar but also an additional (massive) vector; these account for the increase by D in the dimension of the space of level-1 physical states in comparison to the light-cone gauge. Quantization has broken the gauge invariance of the classical theory, thereby introducing additional degrees of freedom that have no classical analog.

Finally, if $D = 26$ the scalar state (5.80) is null; it is also orthogonal to all other allowed states (as a further calculation shows). The same is true of the additional $(D-1)$-vector state, so if we define physical states as equivalence classes (in the way described for level 1) then we recover the level-2 results of the light-cone gauge.

It can be shown that provided $a = 1$ and $D = 26$, the results of the light-cone gauge are recovered in this way at all higher levels.

6. Interlude: Path integrals and the point particle

Let $A(X)$ be the quantum-mechanical amplitude for a particle to go from the origin of Minkowski coordinates to some other point in Minkowski space-time with cartesian coordinates X. As shown by Feynman, $A(X)$ has a path-integral representation. In the case of a relativistic particle of mass m, with phase-space action $I[x, p; e]$ we have

$$A(X) = \int [de] \int [dxdp]\, e^{iI[x,p;e]}\,, \qquad x(0) = 0\,, \quad x(1) = X\,. \tag{6.1}$$

Here we are parametrising the path such that it takes *unit parameter time* to get from the space-time origin to the space-time point with coordinates X. The integrals have still to be defined, but we proceed formally for the moment.

We now allow t to be complex and we "Wick rotate": first set $t = -i\tilde{t}$ to get

$$I[x, p; e] = \int d\tilde{t} \left\{ \dot{x}^m p_m + \frac{i}{2} e\left(p^2 + m^2\right) \right\} \qquad (\dot{x} = dx/d\tilde{t})\,. \tag{6.2}$$

As it stands, $\tilde{t}$ is pure imaginary, but we can rotate the contour in the complex $\tilde{t}$-plane back to the real axis; if we choose to call this real integration variable t then this procedure takes

$$-iI[x, p; e] \to \int dt \left\{ -i\dot{x}^m p_m + \frac{1}{2} e\left(p^2 + m^2\right) \right\} = I_E[x, p; e]\,, \tag{6.3}$$

where I_E is the "Euclidean action". The amplitude A is now given by the Euclidean path integral

$$A(X) = \int [de] \int [dxdp]\, e^{-I_E}\,. \tag{6.4}$$

We will fix the gauge invariance by setting $e = s$ for constant s. As discussed previously, the variable s is gauge-invariant, so it is not possible to use gauge invariance to bring it to a particular value, so we have to integrate over s, which (being proportional to the elapsed proper time) could be any number from zero to infinity. In other words, we can use gauge invariance to reduce the functional integral over $e(t)$ to an ordinary integral over s from zero to infinity. We now have

$$A(x) = \int_0^\infty ds \int [dxdp]\, e^{\int dt \left\{ i\dot{x}^m p_m - \frac{s}{2}\left(p^2 + m^2\right) \right\}}\,. \tag{6.5}$$

This is not quite right, for reasons to be explained soon, but it will suffice for the moment.

To define the $\int [dx\, dp]$ integral we first approximate the path in some way. We could do this by n straight-line segments. We would then have n D-momentum integral to do (one for each segment) and $(n-1)$ integrals over the D-vector positions of the joins. This illustrates the general point that in any multiple-integral approximation to the phase-space path integral there will be some number of phase-space pairs of integrals plus one extra D-momentum integral, which is the average of $p(t)$.

Consider the $n=1$ case, and choose t such that $x(0)=0$ and $x(1)=X$ (we are still free to choose the parameter time interval because a rescaling of it can be compensated by a rescaling of s, which has no effect on the integral over s). In this case

$$x(t) = Xt \quad \& \quad p(t) = P \quad \Rightarrow \quad -I_E = iX^m P_m - \frac{s}{2}\left(P^2 + m^2\right). \qquad (6.6)$$

The only free variable on which the Euclidean action depends is the particle's D-momentum P, so $\int [dx dp]$ is approximated by the momentum-space integral $\int d^D P$, and we find that

$$A_1(X) = \int d^D P\, e^{iX\cdot P} \int_0^\infty ds\, e^{-\frac{s}{2}\left(P^2 + m^2\right)} \propto \int d^D P \frac{e^{iX\cdot P}}{P^2 + m^2}, \qquad (6.7)$$

which is the Fourier transform of the standard momentum-space Feynman propagator for a particle of mass m and zero spin.

That was just the $n=1$ approximation! A simpler alternative to approximation by segments (which differs from it only for $n>1$) is approximation by polynomials, nth order for $x(t)$ and $(n-1)$th order for $p(t)$. Consider the $n=2$ case:

$$x(t) = (X - x_1)\,t + x_1 t^2 \quad \& \quad p(t) = P + 2q\left(t - \frac{1}{2}\right). \qquad (6.8)$$

Notice that $x(t)$ satisfies the b.c.s and P is the integral of $p(t)$ (i.e. the average D-momentum). We have to integrate over the pair (x_1, q) and P. Using these expressions, we find that

$$-I_E = iX \cdot P - \frac{s}{2}\left(P^2 + m^2\right) - \frac{s}{6}p_1^2 - \frac{1}{6s}x_1^2, \qquad (p_1 = q - ix_1/s). \qquad (6.9)$$

This gives

$$A_2(X) = \int d^D P\, e^{iX\cdot P} \int_0^\infty ds\, e^{-\frac{s}{2}\left(P^2 + m^2\right)} \left[\int d^D x_1\, e^{-\frac{x_1^2}{6s}} \int d^D p_1\, e^{-\frac{sp_1^2}{6}} \right]. \qquad (6.10)$$

The bracketed pair of Gaussian integrals is an s-independent constant, so $A_2(X) \propto A_1(X)$. One finds, similarly, that $A_n(X) \propto A_1(X)$. Taking the $n \to \infty$ limit we then have $A(X) \propto A_1(X)$, which is (as we just saw) the Feynman propagator in configuration space.

6.1 Faddeev-Popov determinant

Now we return to the problem of gauge-fixing. The problem with the formula (6.1) is that, because of gauge-invariance, we are integrating over too many functions. Implicitly, we are integrating over functions $\alpha(t)$ that are maps from the (one-dimensional) gauge group to the worldline. If this integral were explicit we could just omit the integral, but it is only implicit, so it is not immediately obvious how we should proceed.

Since we can choose a gauge for which $e(t) = s$, for variable constant s, it must be possible to write an arbitrary function $e(t)$ as a gauge transform of $e = s$:

$$e(t) = s + \dot{\alpha}(t) = e_s\left[\alpha(t)\right] . \tag{6.11}$$

We have now expressed the general $e(t)$ in terms of the gauge group parameter $\alpha(t)$ and the constant s. As a corollary, we have

$$\int [de] = \int_0^\infty ds \int [d\alpha]\, \Delta_{FP} \tag{6.12}$$

where Δ_{FP} is the Jacobian for the change of variables from $e(t)$ to $\{s, \alpha(t)\}$:

$$\Delta_{FP} = \det\left[\frac{\delta e_s[\alpha(t)]}{\delta\alpha(t')}\right] = \det\left[\delta(t - t')\partial_{t'}\right] . \tag{6.13}$$

This is the *Faddeev-Popov determinant*. Using (6.12) in the formula

$$1 = \int [de]\delta[e(t) - s] , \tag{6.14}$$

which defines what we mean by the delta functional, we deduce that

$$1 = \int_0^\infty ds \int [d\alpha]\, \Delta_{FP}\, \delta[e(t) - s] . \tag{6.15}$$

We now return to the initial Euclidean path-integral expression for $A(X)$ and "insert 1" into the integrand; i.e. we insert the RHS of (6.15) into the integrand of (6.4). This gives us

$$A(X) = \int [de] \left[\int_0^\infty ds \int [d\alpha]\, \Delta_{FP}\, \delta[e(t) - s]\right] \int [dxdp] e^{-I_E[x,p;e]} . \tag{6.16}$$

Re-ordering the integrals and using the delta functional to do the $[de]$ integral (this sets $e = s$ elsewhere in the integrand) we get

$$A(X) = \int [d\alpha] \int_0^\infty ds\, \Delta_{FP} \int [dxdp] e^{-I_E[x,p;s]} . \tag{6.17}$$

By these manipulations we have made *explicit* the integral over maps from the gauge group to the worldline, so we can remedy the problem of too many integrals by simply omitting the $[d\alpha]$ integral. This gives us

$$A(X) = \int_0^\infty ds \, \Delta_{FP} \int [dxdp] e^{-I_E[x,p;s]} \, . \tag{6.18}$$

This replaces (6.5), from which the Δ_{FP} factor was missing. That's why (6.5) was "not quite right", but the Δ_{FP} factor only effects the normalisation of $A(X)$, which anyway depends on the detailed definitions of the path integral measures.

Although the FP determinant is not relevant to the computation of $A(X)$ it is relevant to other computations, and it is very important to the path-integral quantization of the NG string, which we will get to soon.

6.2 Fadeev-Popov ghosts

Let (b_i, c^i) $(i = 1, \ldots, n)$ be n pairs of *anticommuting* variables. This means that

$$\{b_i, b_j\} = 0 \, , \quad \{b_i, c^j\} = 0 \, , \quad \{c^i, c^j\} = 0 \quad \forall \, i, j = 1, \ldots, n \, , \tag{6.19}$$

where $\{,\}$ means anticommutator: $\{A, B\} = AB + BA$. Any function of anticommuting variables has a terminating Taylor expansion because no one anticommuting variable can appear twice. Consider the $n = 1$ case

$$f(b, c) = f_0 + b f_1 + c f_{-1} + b c \tilde{f}_0 \, , \tag{6.20}$$

where $(f_0, f_{\pm 1}, \tilde{f}_0)$ are independent of both b and c. Then

$$\frac{\partial}{\partial b} f = f_1 + c \tilde{f}_0 \quad \Rightarrow \quad \frac{\partial}{\partial c} \frac{\partial}{\partial b} f = \tilde{f}_0 \, . \tag{6.21}$$

Essentially, a derivative with respect to b removes the part of f that is independent of b and then strips b off what is left. However, we should move b to the left of anything else before stripping it off; this is equivalent to the definition of the derivative as a "left derivative". Using this definition we have

$$\frac{\partial}{\partial c} f = f_{-1} - b \tilde{f}_0 \quad \Rightarrow \quad \frac{\partial}{\partial b} \frac{\partial}{\partial c} f = -\tilde{f}_0 \, . \tag{6.22}$$

There is minus sign relative to (6.21) because we had to move c to the left of b. This result shows that

$$\left\{ \frac{\partial}{\partial b}, \frac{\partial}{\partial c} \right\} = 0 \, . \tag{6.23}$$

That is, *partial derivatives with respect to anticommuting variables anti-commute*. In particular, since a function of anticommuting variables is necessarily linear in any one of them,

$$\left[\frac{\partial}{\partial b} \right]^2 = 0 \, , \qquad \left[\frac{\partial}{\partial c} \right]^2 = 0 \, . \tag{6.24}$$

We can also integrate over anticommuting variables. The (Berezin) integral over an anticommuting variable is defined to be the same as the partial derivative with respect to it. Consider, for example, the Gaussian integral

$$\int d^n b\, d^n c\ e^{-b_i M^i{}_j c_j} = \left[\frac{\partial}{\partial b_n}\cdots\frac{\partial}{\partial b_1}\right]\left[\frac{\partial}{\partial c^n}\cdots\frac{\partial}{\partial c^1}\right] e^{-b_i M^i{}_j c_j}\ . \tag{6.25}$$

If we expand the integrand in powers of bMc the expansion terminates at the nth term, which is also the only term that contributes to the integral because it is the only one to contain all b_i and all c^i. Because of the anti-commutativity of the partial derivatives, we then find that

$$\int d^n c\, d^n b\ e^{-b_i M^i{}_j c_j}\ \propto\ \frac{1}{n!}\,\varepsilon_{i_1\cdots i_n} M^{i_1}{}_{j_1}\cdots M^{i_n}{}_{j_n}\,\varepsilon^{j_1\cdots j_n} = \det M \tag{6.26}$$

We can use a functional variant of this result to write the FP determinant as a Gaussian integral over anticommuting "worldline fields" $b(t)$ and $c(t)$:

$$\det\left[\delta(t-t')\partial_t\right] = \int [db\, dc]\exp\left[-i\int dt\int dt'\,b(t)\left[\delta(t-t')\partial_{t'}\right]c(t')\right]$$
$$= \int [db\, dc]\exp\left[-i\int dt\,b\dot{c}\right]\ . \tag{6.27}$$

The factor of i in the exponent is not significant; it is just the consequence of a convention (and it is only a convention) that a product of two "real" anticommuting variables is "imaginary", so an i is needed for reality[16]. The anticommuting worldline fields are known collectively as the FP ghosts, although it is useful to distinguish between them by calling c the ghost and b the anti-ghost.

N.B. There is no relation between the FP ghosts and the ghosts that appear in the NG string spectrum for $D > 26$. The same word is being used for two entirely different things!

Using (6.27) in the expression (6.18) we arrive at the result

$$A(X) = \int_0^\infty ds\int[dxdp]\int[db\, dc]\,e^{iI_{qu}}\ , \tag{6.28}$$

where the "quantum" action is

$$I_{qu} = \int dt\,\{\dot{x}^m p_m + ib\dot{c} - H_{qu}\}\ , \qquad H_{qu} = \frac{s}{2}\left(p^2 + m^2\right)\ . \tag{6.29}$$

---

[16]Of course, an anticommuting number cannot really be real; it is "real" if we declare it to be unchanged by complex conjugation. Given two such anti-commuting numbers μ and ν we may construct the complex anti-commuting number $\mu + i\nu$, which will then have complex conjugate $\mu - i\nu$. According to the convention, the product $i\mu\nu$ is real.

We now have a mechanical system with an extended phase-space with additional, anticommuting, coordinates (b, c).

There is a simple extension of these ideas to the general mechanical system with n first class constraints φ_i imposed by Lagrange multipliers λ^i. The FP operator is found by varying the gauge-fixing condition with respect to the gauge parameter, so in the gauge $\lambda^i = \bar{\lambda}^i$, for constants $\bar{\lambda}^i$, we find that

$$\Delta_{FP} = \det \left(\frac{\delta_\epsilon \lambda^i(t')}{\delta \epsilon^j(t)} \bigg|_{\lambda^i = \bar{\lambda}^i} \right) \tag{6.30}$$

where $\delta_\epsilon \lambda^i$ is the gauge variation of λ^i with parameters ϵ^i, given in (2.46). This gives

$$\Delta_{FP} = \det \left[\delta(t - t') \left(\delta^i_j \partial_t + \bar{\lambda}^k f_{jk}{}^i \right) \right] \quad \propto \quad \int [dbdc]\, e^{iI_{FP}[b,c]}, \tag{6.31}$$

where the FP action is

$$I_{FP}[b, c] = \int dt \left\{ ib_i \left[\dot{c}^i + \bar{\lambda}^k c^j f_{jk}{}^i \right] \right\}. \tag{6.32}$$

This must be added to the original action to get the "quantum action"

$$I_{qu} = \int dt \left\{ \dot{q}^I p_i + ib_i \dot{c}^i - H_{qu} \right\}, \qquad H_{qu} = \bar{\lambda}^k \left(\varphi_k + ic^j f_{jk}{}^i b_i \right). \tag{6.33}$$

6.2.1 Phase superspace, and the super-Jacobi identity

We now have an action for a mechanical system with an extended phase space (actually a *superspace*) for which some coordinates are anticommuting. On this space we have the following closed non-degenerate (i.e. invertible) 2-form

$$\Omega = dp_m \wedge dx^m + idb_i \wedge dc^i. \tag{6.34}$$

This is "orthosymplectic" rather than "symplectic" because the anticommutativity of b and c means that[17]

$$db_i \wedge dc^i = dc^i \wedge db_i. \tag{6.35}$$

This leads to a canonical Poisson bracket for b and c that is *symmetric* rather than antisymmetric

$$\left\{ b_i, c^j \right\}_{PB} = \left\{ c^j, b_i \right\}_{PB} = -i\delta^j_i. \tag{6.36}$$

More generally, for any two phase-space functions (f, g) that are either commuting or anticommuting

$$\{f, g\}_{PB} + (-1)^{|f||g|} \{g, f\}_{PB} \equiv 0 \tag{6.37}$$

---

[17]The usual minus sign coming from the antisymmetry of the wedge product of 1-forms is cancelled by the minus sign coming from changing the order of b and c.

where $|f|$ is the Grassmann parity of f; i.e. $|f| = 0$ if f is commuting (even Grassmann parity) and $|f| = 1$ if f is anticommuting (odd Grassmann parity), and similarly for g. This means that the PB of any two functions of definite Grassmann parity is antisymmetric unless both are anticommuting, in which case the PB is symmetric, as in (6.36).

This property means that functions on phase space are no longer elements of a Lie algebra with respect to the Poisson bracket. Instead, they are elements of a Lie superalgebra, which is a bilinear product with the (anti)symmetry properties just described such that any three elements satisfy a super-Jacobi identity. For the PB defined by the inverse of an orthosymplectic 2-form this identity is

$$
\begin{aligned}
\{\{f, g\}_{PB}, h\}_{PB} + \ & (-1)^{|f|(|g|+|h|)} \{\{g, h\}_{PB}, f\}_{PB} \\
& + (-1)^{|h|(|f|+|g|)} \{\{h, f\}_{PB}, g\}_{PB} \equiv 0.
\end{aligned}
\tag{6.38}
$$

This is easy to remember because it is just a cycling of (f, g, h) with a factor of -1 whenever an anticommuting function is "passed through" another anticommuting function.

Quantisation can be achieved as before by applying to the canonical variables a variant of the PB to commutator rule in which the commutator is replaced by the anticommutator for Grassmann odd variables. Thus $\{b_i, c^j\}_{PB} \to -i\{\hat{b}_i, \hat{c}^j\}$, where $\{,\}$ indicates an anticommutator. Applying this prescription to (6.36) we deduce the canonical *anticommutation* relations

$$
b_i c^j + c^j b_i \equiv \left\{ \hat{b}_i, \hat{c}^j \right\} = \delta_i^j.
\tag{6.39}
$$

For clarity, operators now have hats again. We still say that $\hat{b}_i$ and $\hat{c}^j$ are "Grassmann odd" operators even though their anticommutator is no longer zero, and this allows us to assign Grassmann parity to functions involving products of $\hat{b}_i$ and $\hat{c}^j$. The super-Jacobi identity for three operators of definite Grassmann parity now reduces to the standard Jacobi identity, except when at least two of the three have odd Grassman parity. Let E be an even operator and (O, O', O'') odd operators; then

$$
\begin{aligned}
[\{O, O'\}, E] + \{[E, O], O'\} + \{[E, O'], O\} \equiv 0, \\
[\{O, O'\}, O''] + [\{O', O''\}, O] + [\{O'', O\} O'] \equiv 0.
\end{aligned}
\tag{6.40}
$$

These are actually identities for *any* three operators, as is the standard Jacobi identity, *irrespective of whether we consider them even or odd*. However, the above super-Jacobi identities are generally more relevant when some of the canonical variables satisfy canonical anti-commutation relations.

6.3 BRST invariance[*]

The "quantum" point particle action (6.29) is invariant under the following transformations with constant anticommuting parameter Λ:

$$\delta_\Lambda x = i\Lambda c p, \quad \delta_\Lambda p = 0, \qquad \delta_\Lambda b = -\frac{1}{2}\left(p^2 + m^2\right)\Lambda, \quad \delta_\Lambda c = 0. \tag{6.41}$$

Notice that the transformations of (x, p) are gauge transformations with parameter $\alpha(t) = -i\Lambda c(t)$; the factor of i is included because Λ is assumed to anticommute with c and we use the convention that complex conjugation changes the order of anticommuting numbers, so we need an i for "reality". Allowing Λ to be t-dependent we find [Exercise: check this]

$$\delta_\Lambda I_{qu} = \frac{i}{2}\int dt\, \dot{\Lambda}\left(p^2 + m^2\right)c. \tag{6.42}$$

This confirms the invariance for constant Λ and tells us that the (anti-commuting) Noether charge is

$$Q_{BRST} = \frac{1}{2}c\left(p^2 + m^2\right). \tag{6.43}$$

This is the BRST charge (named after Becchi, Rouet, Stora and Tyutin, who discovered BRST symmetry in the context of Yang-Mills theory). It generates the BRST transformations (6.41) via Poisson brackets defined for functions on the extended phase space. Let's check this:

$$\delta_\Lambda x = \{x, i\Lambda Q_{BRST}\}_{PB} = i\Lambda c\left\{x, \frac{1}{2}p^2\right\}_{PB} = i\Lambda c p$$

$$\delta_\Lambda b = \{b, i\Lambda Q_{BRST}\}_{PB} = -\frac{i}{2}\Lambda\{b, c\}_{PB}\left(p^2 + m^2\right) = -\frac{1}{2}\left(p^2 + m^2\right)\Lambda. \tag{6.44}$$

BRST symmetry seems rather mysterious, but it's not just a special feature of the point particle. Consider the quantum action (6.33) of the general mechanical model with first class constraints (which includes, as we have seen, the NG string). If we assume that the constraint functions φ_i span a *Lie algebra* then the structure functions f_{ij}^k will be constants satisfying (as a consequence of the Jacobi identity)

$$f_{[ij}{}^\ell f_{k]\ell}{}^m \equiv 0. \tag{6.45}$$

In this case[18] the action is invariant under the transformations generated by

$$Q_{BRST} = c^i\varphi_i + \frac{i}{2}c^i c^k f_{ki}{}^j b_j, \tag{6.46}$$

which satisfies

$$\{Q_{BRST}, Q_{BRST}\}_{PB} = 0. \tag{6.47}$$

---

[18]There is still a BRST charge if the structure functions are *not* constants, but it is more complicated.

This is not a trivial property of Q_{BRST} because the PB is *symmetric* under exchange of its two arguments if these are both anticommuting.

We still need to check the invariance. Because the variation of $(\dot{q}^I p_I + ib\dot{c})$ is guaranteed to be a total time derivative by an extension to the extended phase-space of the lemma summarised by (2.42), we need only check that

$$0 = \{Q_{BRST}, H_{qu}\}_{PB} , \qquad H_{qu} = \bar{\lambda}^k \left(\varphi_k + ic^j f_{jk}{}^i b_i\right) . \tag{6.48}$$

This can be checked directly, but an alternative is to first check that

$$H_{qu} = i \left\{\bar{\lambda}^i b_i, Q_{BRST}\right\}_{PB} , \tag{6.49}$$

and then use the super-Jacobi identity to show that

$$\left\{Q_{BRST}, \left\{\bar{\lambda}^i b_i, Q_{BRST}\right\}_{PB}\right\}_{PB} = \frac{1}{2} \left\{\{Q_{BRST}, Q_{BRST}\}_{PB} , \bar{\lambda}^i b_i\right\}_{PB} , \tag{6.50}$$

which is zero by (6.47).

6.3.1 BRST Quantization*

Consider first the general model with "quantum" action (6.33). The canonical commutation relations for the extended phase-space coordinates are (hats restored, for clarity)

$$[\hat{x}^m, \hat{p}_n] = i\delta_n^m , \qquad \{\hat{b}_i, \hat{c}^j\} = \delta_i^j . \tag{6.51}$$

In addition,

$$\{\hat{b}_i, \hat{b}_j\} = 0 \qquad \left\{\hat{c}^i, \hat{c}^j\right\} = 0 . \tag{6.52}$$

The "quantum" Hamiltonian now becomes the operator

$$\hat{H}_{qu} = \bar{\lambda}^k \left(\hat{\varphi}_i + i\hat{c}^j f_{jk}{}^i \hat{b}_i\right) . \tag{6.53}$$

Notice that $\hat{H}$ commutes with the *ghost number* operator

$$n_{gh} = \hat{c}^i \hat{b}_i . \tag{6.54}$$

This operator also has the property that

$$\left[n_{gh}, \hat{c}^i\right] = \hat{c}^i \qquad \left[n_{gh}, \hat{b}_i\right] = -\hat{b}_i , \tag{6.55}$$

so the c-ghosts have ghost number 1 and the b-ghosts have ghost number -1.

The BRST charge becomes the following operator of unit ghost number:

$$\hat{Q}_{BRST} = \hat{c}^i \hat{\varphi}_i + \frac{i}{2} \hat{c}^i \hat{c}^k f_{ki}{}^j \hat{b}_j . \tag{6.56}$$

Assuming that the PB to (anti)commutator rule applies, we learn from (6.47) that

$$\hat{Q}_{BRST}^2 = 0 \,. \tag{6.57}$$

This is the fundamental property of the BRST charge. As a consequence of this property, it is consistent to impose the physical state condition

$$\hat{Q}_{BRST}|\text{phys}\rangle = 0 \,. \tag{6.58}$$

This condition has the following motivation. The Hamiltonian operator $\hat{H}$ is gauge-dependent[19] so its matrix elements cannot be physical. However, the quantum version of (6.49) is

$$\hat{H} = \left[\bar{\lambda}^i \hat{b}_i, \hat{Q}_{BRST} \right] \,, \tag{6.59}$$

so it follows from (6.58) that, for any two physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$,

$$\langle \text{phys}'|\hat{H}|\text{phys}\rangle = 0 \,. \tag{6.60}$$

In other words, the BRST physical state condition ensures that all physical matrix elements are gauge-independent.

The "physical state" condition (6.58) does not actually remove all unphysical states because for any state $|\chi\rangle$ the state $\hat{Q}_{BRST}|\chi\rangle$ will be "physical", by the definition (6.58), as a consequence of the nilpotency of Q_{BRST}, but it will also be null if we assume an inner product for which $\hat{Q}_{BRST}$ is hermitian:

$$
\begin{aligned}
\left\| \hat{Q}_{BRST}|\chi\rangle \right\|^2 &= \langle \chi|\hat{Q}_{BRST}^\dagger \hat{Q}_{BRST}|\chi\rangle \\
&= \langle \chi|\hat{Q}_{BRST}^2|\chi\rangle \qquad \left(\text{if } \hat{Q}_{BRST}^\dagger = \hat{Q}_{BRST} \right) \\
&= 0
\end{aligned}
\tag{6.61}
$$

So we should really define physical states as equivalence classes (cohomology classes of $\hat{Q}_{BRST}$), where the equivalence relation is

$$|\Psi\rangle \sim |\Psi\rangle + \hat{Q}_{BRST}|\chi\rangle \tag{6.62}$$

for any state $|\chi\rangle$. This is consistent because $\hat{Q}_{BRST}|\chi\rangle$ is orthogonal to *all* states that are physical by the definition (6.58).

Let's now see how these ideas apply to the point particle. In a basis for which $(\hat{x}, \hat{c})$ are diagonal, with eigenvalues (x, c), the the canonical (anti)commutation relations are realised by the operators

$$\hat{p}_m = -i\partial_m \,, \qquad \hat{b} = \frac{\partial}{\partial c} \,, \tag{6.63}$$

---

[19]Although the constants $\bar{\lambda}^i$ may be gauge invariant if the gauge condition $\lambda^i = \bar{\lambda}^i$ is assumed to hold for all t, these constants could still be changed locally. The main point is that different gauge-fixing conditions lead to a different $\hat{H}$.

acting on wavefunctions $\Psi(x, c)$, which we can expand as

$$\Psi(x, c) = \psi_0(x) + c\psi_1(x) \,. \tag{6.64}$$

The BRST charge is now

$$\hat{Q}_{BRST} = -\frac{1}{2}c\left(\Box - m^2\right) \,, \tag{6.65}$$

so the physical state condition is $c(\Box - m^2)\psi_0(x) = 0$, which implies that ψ_0 is a solution to the Klein-Gordon equation. We learn nothing about $\psi_1(x)$, but the equivalence relation (6.62) tells us that

$$\psi_1 \sim \psi_1 + \left(\Box - m^2\right)\chi_0 \,, \tag{6.66}$$

for any function $\chi_0(x)$, which implies that ψ_1 is equivalent to zero unless it too is a solution of the KG equation[20]. So we actually get a doubling of the expected physical states (solutions of the KG equation). For this reason, we have to impose the additional condition

$$b\,|\text{phys}\rangle = 0 \qquad \Rightarrow \quad \psi_1 = 0 \,. \tag{6.67}$$

All this depends on a choice of inner product for which $\hat{Q}_{BRST}$ is hermitian, despite being nilpotent. This is achieved in the point particle case by the choice

$$\langle\Psi|\Psi'\rangle = \int d^D x \int dc\, \Psi^*\Psi' = \int d^D x\, \frac{\partial}{\partial c}\left[\Psi^*\Psi'\right] \,. \tag{6.68}$$

With respect to this inner product, the operators $\hat{b}$ and $\hat{c}$ are hermitian, and hence $\hat{Q}_{BRST}$ is hermitian. Using this inner product we can construct a *field theory* action from which the BRST physical state condition emerges as a field equation. This action is

$$\begin{aligned}
S[\Psi(x, c)] &= \langle\Psi|\hat{Q}_{BRST}|\Psi\rangle \\
&= \frac{1}{2}\int d^D x \int \frac{\partial}{\partial c}\left[\psi_0 c\left(\Box - m^2\right)\psi_0\right] \\
&= \frac{1}{2}\int d^D x\, \psi_0\left(\Box - m^2\right)\psi_0 \,,
\end{aligned} \tag{6.69}$$

which is the Klein-Gordon action.

---

[20]Expand both sides in terms of eigenfunctions of the KG operator, and compare coefficients; all coefficients in the expansion of $(\Box - m^2)\chi_0$ are arbitrary except the coefficients of zero modes of the KG operator, which are zero.

7. Path integrals and the NG string

We now aim to use the conformal gauge in a path-integral approach to quantisation of the closed NG string. Recall that the conformal gauge for the NG string is $\lambda^{\pm} = 1$, and that in this gauge the gauge variation of $\lambda^{\pm}$ is

$$\delta\lambda^{\pm} = \mp\sqrt{2}\,\partial_{\mp}\xi^{\pm}\,. \tag{7.1}$$

The FP determinant is therefore

$$\Delta_{FP} = \det\begin{pmatrix} \sqrt{2}\,\delta(t-t')\delta(\sigma-\sigma')\partial'_{+} & 0 \\ 0 & -\sqrt{2}\,\delta(t-t')\delta(\sigma-\sigma')\partial'_{-} \end{pmatrix}\,, \tag{7.2}$$

so the FP-ghost contribution to the action is[21]

$$I_{FP} = \sqrt{2}\,i\int dt\oint d\sigma\left\{b\,\partial_{+}c - \tilde{b}\,\partial_{-}\tilde{c}\right\}\,. \tag{7.3}$$

This action is conformal invariant, with the ghosts transforming in the same way as the parameters under a composition of two transformations; the parameters $\xi^{\pm}$ are the light-cone components of a worldsheet vector field so the ghosts $(c,\tilde{c})$ will transform as a vector field. Invariance of the action then requires $(b,\tilde{b})$ to transform as the light-cone components of a (symmetric traceless) "quadratic differential". For example,

$$\delta_{\xi}c = \xi^{-}\partial_{-}c - \left(\partial_{-}\xi^{-}\right)c\,, \qquad \delta_{\xi}b = \xi^{-}\partial_{-}b + 2\left(\partial_{-}\xi^{-}\right)b\,, \tag{7.4}$$

and similarly for $(\tilde{c},\tilde{b})$. The Noether charges are the non-zero components of the FP-ghost stress tensor. These are

$$\Theta^{(gh)}_{--} = -\frac{i}{\sqrt{2}}\left[2b\,\partial_{-}c - c\,\partial_{-}b\right]\,, \qquad \Theta^{(gh)}_{++} = \frac{i}{\sqrt{2}}\left[2\tilde{b}\,\partial_{+}\tilde{c} - \tilde{c}\,\partial_{+}\tilde{b}\right]\,. \tag{7.5}$$

Using the ghost equations of motion, in the form $\dot{c} = -c'$ etc., these become

$$\Theta^{(gh)}_{--} = -i\left[2b\,c' + bc'\right]\,, \qquad \Theta^{(gh)}_{++} = i\left[2\tilde{b}\,\tilde{c}' + \tilde{b}\,\tilde{c}'\right]\,. \tag{7.6}$$

Adding the FP action to the usual phase-space action, we get the "quantum" action for the closed NG string in conformal gauge

$$I_{qu}[X, P; b, c; \tilde{b}, \tilde{c}] = \int dt\oint d\sigma\left\{\dot{X}^{m}P_{m} + ib\dot{c} + i\tilde{b}\dot{\tilde{c}} - \mathcal{H}_{qu}\right\}\,,$$

$$\mathcal{H}_{qu} = \frac{P^{2}}{2T} + \frac{T}{2}(X')^{2} - i\left(bc' - \tilde{b}\tilde{c}'\right)\,. \tag{7.7}$$

---

[21]It is because the conformal gauge fails to completely fix the gauge invariance that we get a non-trivial FP action. Whenever the gauge is fixed completely, the FP action is one for which the FP ghosts can be trivially eliminated; that's not the case here because we can't invert $\partial_{\pm}$, and that is also the reason that there is a residual conformal symmetry. From this fact, one can see that the FP ghosts are (in a sense made precise by the BRST formalism) subtracting out the residual unphysical degrees of freedom that survive the conformal gauge condition.

From this action we can read off the PB relations; in particular

$$\{b(\sigma), c(\sigma')\}_{PB} = -i\delta(\sigma - \sigma') = \{\tilde{b}(\sigma), \tilde{c}(\sigma')\}_{PB}. \tag{7.8}$$

We now pass to the Fourier-mode form of the action. In addition to the Fourier series for $P \pm TX'$, we will need the Fourier series expansions

$$c = \sum_{k\in\mathbb{Z}} e^{ik\sigma} c_k, \qquad b = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{ik\sigma} b_k,$$

$$\tilde{c} = \sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{c}_k, \qquad \tilde{b} = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{b}_k. \tag{7.9}$$

The "quantum" action in terms of Fourier modes is[22]

$$I_{qu} = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \left(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k} \right) + \sum_{n\in\mathbb{Z}} i \left(b_{-n}\dot{c}_n + \tilde{b}_{-n}\dot{\tilde{c}}_n \right) - H_{qu} \right\},$$

$$H_{qu} = \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \tag{7.10}$$

where

$$\mathcal{L}_0 = L_0 + N_{(gh)} = \frac{1}{2}\alpha_0^2 + N_{qu}, \qquad N_{qu} = N + N_{(gh)},$$

$$\tilde{\mathcal{L}}_0 = \tilde{L}_0 + N_{(gh)} = \frac{1}{2}\alpha_0^2 + \tilde{N}_{qu}, \qquad \tilde{N}_{qu} = \tilde{N} + \tilde{N}_{(gh)}. \tag{7.11}$$

The ghost level numbers are

$$N_{(gh)} = \sum_{k=1}^{\infty} k \left(b_{-k}c_k + c_{-k}b_k \right), \qquad \tilde{N}_{(gh)} = \sum_{k=1}^{\infty} k(\tilde{b}_{-k}\tilde{c}_k + \tilde{c}_{-k}\tilde{b}_k). \tag{7.12}$$

We can now read off the PBs of the Fourier modes. For the new, anticommuting, variables we have

$$\{c_n, b_{-n}\}_{PB} = -i, \qquad \left\{ \tilde{c}_n, \tilde{b}_{-n} \right\}_{PB} = -i, \qquad (n \in \mathbb{Z}). \tag{7.13}$$

Notice that $n = 0$ is included, although the (anti)ghost zero modes (b_0, c_0) and $(\tilde{b}_0, \tilde{c}_0)$ do not appear in the Hamiltonian. These anti-commutation relations are equivalent to (7.8).

The Fourier modes of the ghost/anti-ghost stress tensor components of (7.6) are

$$L_m^{(gh)} = \oint d\sigma\, e^{-im\sigma} \Theta_{--}^{(gh)} = \sum_{k\in\mathbb{Z}} (m + k)\, b_{m-k}c_k,$$

$$\tilde{L}_m^{(gh)} = \oint d\sigma\, e^{im\sigma} \Theta_{++}^{(gh)} = \sum_{k\in\mathbb{Z}} (m + k)\, \tilde{b}_{m-k}\tilde{c}_k. \tag{7.14}$$

---

[22]For the open string with free ends just omit all variables with a tilde. In this case the centre of mass "quantum" action is precisely that of the point particle.

Notice that
$$L_0^{(gh)} = N_{(gh)}, \qquad \tilde{L}_0^{(gh)} = \tilde{N}_{(gh)}. \tag{7.15}$$

Using the PB relations (7.13) one finds that
$$\left\{L_m^{(gh)}, c_n\right\}_{PB} = i\,(2m+n)\,c_{n+m}, \qquad \left\{L_m^{(gh)}, b_n\right\}_{PB} = -i\,(m-n)\,b_{m+n} \tag{7.16}$$

and this leads to
$$\left\{L_m^{(gh)}, L_n^{(gh)}\right\}_{PB} = -i\,(m-n)\,L_{m+n}^{(gh)}, \tag{7.17}$$

and similarly for $\tilde{L}_m^{(gh)}$.

The total "quantum" conformal charges are
$$\mathcal{L}_m = L_m + L_m^{(gh)}, \qquad \tilde{\mathcal{L}}_m = \tilde{L}_m + \tilde{L}_m^{(gh)}. \tag{7.18}$$

These are the Fourier modes of the non-zero components of the energy-momentum stress tensor of the quantum action (7.7), and their algebra is that of Witt $\oplus$ Witt:
$$\{\mathcal{L}_m, \mathcal{L}_n\}_{PB} = -i\,(m-n)\,\mathcal{L}_{m+n}$$
$$\{\mathcal{L}_m, \tilde{\mathcal{L}}_n\}_{PB} = 0$$
$$\{\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n\}_{PB} = -i\,(m-n)\,\tilde{\mathcal{L}}_{m+n}. \tag{7.19}$$

The PBs of the $\mathcal{L}_m$ with the Fourier modes of the various fields determine the transformations of these fields under the residual conformal invariance of the conformal gauge. In general, for any phase-space function $\mathcal{O}^{(J)}$ of conformal dimension J,
$$\left\{\mathcal{L}_m, \mathcal{O}_n^{(J)}\right\}_{PB} = -i\,[m\,(J-1)-n]\,\mathcal{O}_{n+m}^{(J)}. \tag{7.20}$$

For the canonical variables we have
$$\{\mathcal{L}_m, \alpha_k\}_{PB} = ik\alpha_{k+m},$$
$$\{\mathcal{L}_m, c_k\}_{PB} = i\,(2m+k)\,c_{k+m},$$
$$\{\mathcal{L}_m, b_k\}_{PB} = -i\,(m-k)\,b_{k+m}. \tag{7.21}$$

This tells us that $\partial_- X$ has conformal dimension 1 (since $P - TX' = -\sqrt{2}\,T\,\partial_- X$ when $P = T\dot{X}$) while c has conformal dimension -1 and b has conformal dimension 2. We should also consider the transformations generated by $\tilde{\mathcal{L}}_m$; these are entirely analogous, so all canonical variables actually have two conformal dimensions. For example
$$[\partial_- X] = (1,0), \qquad [\partial_+ X] = (0,1), \tag{7.22}$$

where $[\,]$ denotes "conformal dimensions" of the bracketed quantity. These are the conformal dimensions of the light-cone components of a worldsheet 1-form. Compare this with
$$[c] = (-1,0), \qquad [\tilde{c}] = (0,-1), \tag{7.23}$$

which are the conformal dimensions of the light-cone components of a worldsheet vector. Finally,

$$[b] = (2, 0), \qquad \left[\tilde{b}\right] = (0, 2).$$

(7.24)

After elimination of P from the action (7.7), we get the generalisation of the conformal gauge action (3.84):

$$I_{qu} = -T \int d^2\sigma \left\{ \partial_+ X \cdot \partial_- X - \frac{\sqrt{2}}{T} i \left(b \, \partial_+ c - \tilde{b} \, \partial_- \tilde{c} \right) \right\}.$$

(7.25)

From the above results for conformal dimensions, and taking into account the fact acting with ∂_- raises the conformal dimensions by $(1, 0)$ and acting with ∂_+ raises them by $(0, 1)$, we see that *all terms in the Lagrangian have conformal dimension* $(1, 1)$, which is what is required for invariance of the action.

This is all classical, but it carries over to the quantum theory; in particular the conformal dimensions of $P \pm T X'$ and (b, c) are as above. However, whereas the product $\mathcal{O}^{(J)} \mathcal{O}^{(J')}$ of phase-space functions has conformal dimension $J + J'$, this will **not** generally be true in the quantum theory; products of operators can have *anomalous* conformal dimensions.

7.0.2 Critical dimension again

We already know that the algebra satisfied by the L_m is the Virasoro algebra with central charge D, i.e.

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{D}{12} \left(m^3 - m \right) \delta_{m+n}.$$

(7.26)

Now we need to determine the algebra satisfied by the $L_m^{(gh)}$, which must take the form

$$\left[L_m^{(gh)}, L_n^{(gh)} \right] = (m - n) \left(L_{m+n}^{(gh)} - a \delta_{m+n} \right) + \frac{c_{gh}}{12} \left(m^3 - m \right) \delta_{m+n}.$$

(7.27)

for some constants a and c_{gh}. Setting $n = -m$ and taking the oscillator vacuum expectation value of both sides, we deduce (using the fact that $L_m^{(gh)} |0\rangle = 0$ for $m \geq 0$)

$$\left\| L_{-m}^{(gh)} |0\rangle \right\|^2 = -2ma + \frac{c_{gh}}{12} \left(m^3 - m \right).$$

(7.28)

Choosing $m = 1$ and $m = 2$ we deduce that

$$\left\| L_{-1}^{(gh)} |0\rangle \right\|^2 = -2a, \qquad \left\| L_{-2}^{(gh)} |0\rangle \right\|^2 = -4a + \frac{c_{gh}}{2}.$$

(7.29)

Now we use[23]

$$b_k |0\rangle = c_k |0\rangle = 0 \qquad k > 0,$$

(7.30)

---

[23] We will not need to know how b_0 or c_0 act on $|0\rangle$.

to deduce that

$$L_{-1}^{(gh)}|0\rangle = -(b_{-1}c_0 + 2b_0c_{-1})|0\rangle,$$
$$L_{-2}^{(gh)}|0\rangle = -(b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2})|0\rangle. \tag{7.31}$$

Next, we use[24]

$$b_k^\dagger = b_{-k}, \qquad c_k^\dagger = c_{-k} \tag{7.32}$$

to compute the left-hand sides of (7.29). For example,

$$\left\|L_{-1}^{(gh)}|0\rangle\right\|^2 = \langle 0|(c_0b_1 + 2c_1b_0)(b_{-1}c_0 + 2b_0c_{-1})|0\rangle$$
$$= -2\langle 0|(c_0b_0b_1c_{-1} + b_0c_0c_1b_{-1})|0\rangle \quad (\text{using } b_0^2 = c_0^2 = 0)$$
$$= -2\langle 0|(c_0b_0\{b_1, c_{-1}\} + \{c_1, b_{-1}\}b_0c_0)|0\rangle \quad (\text{using } b_1|0\rangle = c_1|0\rangle = 0)$$
$$= -2\langle 0|\{c_0, b_0\}|0\rangle = -2, \tag{7.33}$$

from which we conclude that $a = 1$. Similarly,

$$\left\|L_{-2}^{(gh)}|0\rangle\right\|^2 = \langle 0|(2c_0b_2 + 3c_1b_1 + 4c_2b_0)(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2})|0\rangle$$
$$= -8\langle 0|(c_0b_0b_2c_{-2} + b_0c_0c_2b_{-2})|0\rangle - 9\langle 0|c_1b_{-1}b_1c_{-1}|0\rangle$$
$$= -8\langle 0|(c_0b_0\{b_2, c_{-2}\} + b_0c_0\{c_2, b_{-2}\})|0\rangle - 9\langle 0|\{c_1, b_{-1}\}\{b_1, c_{-1}\}|0\rangle$$
$$= -8\langle 0|(\{c_0, b_0\}|0\rangle - 9 = -17, \tag{7.34}$$

from which we conclude that

$$-4 + \frac{c_{gh}}{2} = -17 \quad \Rightarrow \quad c_{gh} = -26. \tag{7.35}$$

Using these values for a and c_{gh} in (7.27) we find that

$$\left[L_m^{(gh)}, L_n^{(gh)}\right] = (m - n)\left(L_{m+n}^{(gh)} - \delta_{m+n}\right) - \frac{26}{12}\left(m^3 - m\right)\delta_{m+n}, \tag{7.36}$$

and combining this with (7.26), we deduce that

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\left(\mathcal{L}_{m+n} - \delta_{m+n}\right) + \frac{D - 26}{12}\left(m^3 - m\right). \tag{7.37}$$

This is a Virasoro algebra with central charge $D - 26$, which is zero iff $D = 26$.

In conclusion, we find agreement with the light-cone gauge result that $a = 1$ and $D = 26$.

---

[24]This implies that b_0 and c_0 are hermitian, despite the fact that $b_0^2 = c_0^2 = 0$. This is possible for a particular choice of inner product on the two-state space on which these operators act.

7.1 BRST for NG string[*]

The BRST charge can be written as

$$Q_{BRST} = Q_- + Q_+ \,, \tag{7.38}$$

and it follows from the general formula, using the fact that the algebra of constraint functions is $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$, that

$$Q_- = \oint d\sigma \left\{ c \mathcal{H}_- + i c c' b \right\} \,, \qquad Q_+ = \oint d\sigma \left\{ \tilde{c} \mathcal{H}_+ - i \tilde{c} \tilde{c}' \tilde{b} \right\} \,. \tag{7.39}$$

Using the PB relations obeyed by $\mathcal{H}_\pm$, given in (3.24), it is not difficult to verify that

$$\left. \begin{array}{l} \{Q_\pm, Q_\pm\}_{PB} = 0 \\ \{Q_+, Q_-\}_{PB} = 0 \end{array} \right\} \quad \Rightarrow \quad \{Q_{BRST}, Q_{BRST}\}_{PB} = 0 \,. \tag{7.40}$$

A Fourier decomposition of $Q_\pm$ yields the result

$$Q_- = \sum_{n \in \mathbb{Z}} c_{-n} L_n - \frac{1}{2} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} (p - q) \, c_{-p} c_{-q} b_{p+q} \,,$$

$$Q_+ = \sum_{n \in \mathbb{Z}} \tilde{c}_{-n} \tilde{L}_n - \frac{1}{2} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} (p - q) \, \tilde{c}_{-p} \tilde{c}_{-q} \tilde{b}_{p+q} \,. \tag{7.41}$$

Let us rewrite these expressions as

$$Q_- = \sum_{m \in \mathbb{Z}} \left[L_m + \frac{1}{2} L_m^{(gh)} \right] c_{-m} \,, \qquad Q_+ = \sum_{m \in \mathbb{Z}} \left[\tilde{L}_m + \frac{1}{2} \tilde{L}_m^{(gh)} \right] c_{-m} \,, \tag{7.42}$$

A check on these expressions is to verify that

$$\left. \begin{array}{l} \mathcal{L}_0 = i \{b_0, Q_-\}_{PB} \\ \tilde{\mathcal{L}}_0 = i \{\tilde{b}_0, Q_+\}_{PB} \end{array} \right\} \quad \Rightarrow \quad H_{qu} = i \left\{ \left(b_0 + \tilde{b}_0 \right), Q_{BRST} \right\}_{PB} \,, \tag{7.43}$$

which is the formula (6.49) for our case. It may be verified that

$$\mathcal{L}_m = i \{b_m, Q_-\}_{PB} \,, \qquad \tilde{\mathcal{L}}_m = i \{\tilde{b}_m, Q_+\}_{PB} \,. \tag{7.44}$$

Passing to the quantum theory, the non-zero anticommutators of the (anti)ghost modes are

$$\{c_n, b_{-n}\} = 1 \,, \qquad \{\tilde{c}_n, \tilde{b}_{-n}\} = 1 \,. \tag{7.45}$$

We define the (anti)ghost oscillator vacuum as the state

$$|0\rangle_{gh} = |0\rangle_R^{gh} \otimes |0\rangle_L^{gh} \,, \tag{7.46}$$

such that

$$c_n |0\rangle_R^{gh} = 0 \,, \quad n > 0 \qquad \& \qquad b_n |0\rangle_R^{gh} = 0 \,, \quad n \geq 0 \,. \tag{7.47}$$

and similarly for the tilde operators acting on $|0\rangle^{gh}_L$. We can act on this with the (anti)ghost creation operators c_0 and (c_{-n}, b_{-n}) for $n > 0$ to get states in an (anti)ghost Fock space. The full oscillator vacuum is now the tensor product state[25]

$$|0) = |0\rangle \otimes |0\rangle_{gh} \,. \tag{7.48}$$

We should first deal with some operator ordering ambiguities in expressions involving (anti)ghost operators. There is no ambiguity in the expression for $L^{(gh)}_m$ as long as $m \neq 0$; for $m = 0$ we choose the ordering given in (7.12), which ensures that

$$L^{(gh)}_0 \equiv N_{(gh)} = \sum_{k=1}^{\infty} k \left(b_{-k} c_k + c_{-k} b_k \right) \quad \Rightarrow \quad N_{(gh)} |0\rangle_{gh} = 0 \,. \tag{7.49}$$

We then have

$$L^{(gh)}_m |0\rangle_{gh} = 0 \,, \quad m \geq 0 \,. \tag{7.50}$$

Notice that

$$\left[N_{(gh)}, c_{-k} \right] = k c_{-k} \,, \qquad \left[N_{(gh)}, b_{-k} \right] = k b_{-k} \,, \tag{7.51}$$

so that acting with either the ghost creation operator c_{-k} or the anti-ghost creation operator b_{-k} increases the ghost level number by k. It follows that the eigenvalues of $N_{(gh)}$ and $\tilde{N}_{(gh)}$ are the non-negative integers.

There is also an ordering ambiguity in Q_- that allows us to add to it any multiple of $\hat{c}_0$; we choose the order such that

$$Q_- = \left(L_0 + \frac{1}{2} N_{(gh)} - a \right) \hat{c}_0 + \sum_{m=1}^{\infty} \left[\left(L_{-m} + \frac{1}{2} L^{(gh)}_{-m} \right) c_m + c_{-m} \left(L_m + \frac{1}{2} L^{(gh)}_m \right) \right] \,, \tag{7.52}$$

for some constant a. This definition is such that

$$Q_- |0) = 0 \quad \Rightarrow \quad (L_0 - a) |0) = 0 \,. \tag{7.53}$$

It is also such that [Exercise: check this]

$$\{ b_m, Q_- \} = \mathcal{L}_m - a \delta_m \,, \tag{7.54}$$

which is the quantum version of (7.44). Let's also record here that

$$[\mathcal{L}_m, b_n] = (m - n) b_{n+m} \,, \tag{7.55}$$

which is the statement that the operator b has conformal dimension 2.

Now we show how the Virasoro anomaly in the algebra of the $\mathcal{L}_n$ is related to a BRST anomaly. Using (7.54) we find that

$$
\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= [\{b_m, Q_-\}, \mathcal{L}_n] \\
&= -\{[\mathcal{L}_n, b_m], Q_-\} + \{[Q_-, \mathcal{L}_n], b_m\} \\
&= (m - n)\{b_{m+n}, Q_-\} + \{[Q_-, \mathcal{L}_n], b_m\},
\end{aligned}
\tag{7.56}
$$

where the second line follows from the super-Jacobi identity, and the last line uses (7.55). Now we use (7.54) again, and again the super-Jacobi identity, to show that

$$
[Q_-, \mathcal{L}_n] = [Q_-, \{b_n, Q_-\}] = [Q_-^2, b_n] .
\tag{7.57}
$$

Using this in (7.56) we deduce that

$$
[\mathcal{L}_m, \mathcal{L}_n] = (m - n)(\mathcal{L}_{m+n} - a\delta_{m+n}) + \{[Q_-^2, b_n], b_m\} .
\tag{7.58}
$$

This shows that $Q_-^2 = 0$ implies no Virasoro anomaly (i.e. zero central charge c). If Q_-^2 is non-zero it will be some expression quadratic in oscillator operators (the classical result ensures that the quartic term cancels) and it must have ghost number 2, so

$$
Q_-^2 = \frac{1}{2}\sum_{k\in\mathbb{Z}} c_k c_{-k} A(k)
\tag{7.59}
$$

for some function $A(k)$. We then find that

$$
[\mathcal{L}_m, \mathcal{L}_n] = (m - n)(\mathcal{L}_{m+n} - a\delta_{m+n}) + A(m)\delta_{m+n}
\tag{7.60}
$$

This shows that no Virasoro anomaly implies $Q_-^2 = 0$. The same argument applies to Q_+, so we now see that

$$
Q_{BRST}^2 = 0 \quad \Leftrightarrow \quad A(m) = 0 .
\tag{7.61}
$$

In other words, *the absence of a BRST anomaly is equivalent to the vanishing of the central charge in the Virasoro algebra spanned by the $\mathcal{L}_m$ is zero*, so that

$$
[\mathcal{L}_m, \mathcal{L}_n] = (m - n)(\mathcal{L}_{m+n} - a\delta_{m+n}) .
\tag{7.62}
$$

Given that $Q_{BRST}^2 = 0$, it is consistent to impose the BRST physical-state condition $Q_{BRST}|\text{phys}\rangle = 0$. The physical states are then cohomology classes of Q_{BRST}. Consider the state

$$
|\Psi)_R \equiv |\Psi\rangle_R \otimes |0\rangle_R^{gh} ,
\tag{7.63}
$$

where $|\Psi\rangle_R$ is a state in the α-oscillator Fock space. Then

$$
Q_-|\Psi)_R = (L_0 - 1)|\Psi\rangle_R \otimes |0\rangle_R^{gh} + \sum_{m=1}^{\infty} L_m|\Psi\rangle_R \otimes c_{-m}|0\rangle_R^{gh} ,
\tag{7.64}
$$

which is zero only if $|\Psi\rangle_R$ satisfies

$$
(L_0 - 1)|\Psi\rangle_R = 0 \quad \& \quad L_m|\Psi\rangle_R = 0 \quad \forall\, m > 0 .
\tag{7.65}
$$

These are the Virasoro conditions of the "old covariant" method of quantization.

8. Interactions

So far we have seen that each excited state of a NG string can be viewed as a particle with a particular mass and spin. Now we are going to see that the stringy origin of these particles leads naturally to interactions between them. We shall explore this in the context of a path-integral quantization for the closed string.

Consider a closed string propagating from the infinite past to the infinite future; its worldsheet is then an infinite cylinder. Now set $t = -i\tau$ and "Wick-rotate" to make τ real, as described for the particle. Then

$$\sigma^- \to \frac{1}{\sqrt{2}} (\sigma + i\tau) = z, \qquad \sigma^+ \to \bar{z}. \tag{8.1}$$

The worldsheet is now the strip in the complex z-plane defined by

$$-\infty < \tau < \infty, \qquad 0 \le \sigma < 2\pi. \tag{8.2}$$

For this to make sense, we should accompany the worldsheet Wick rotation with a spacetime Wick rotation $(X^0 \to -iX^0)$ so that the spacetime metric is now Euclidean. The induced worldsheet metric is then conformal to the standard Euclidean metric on the z-plane, and *equal to it in conformal gauge.*

In conformal gauge, and after performing the Gaussian path-integral over P (which affects only the overall normalisation), the Euclidean path integral will be weighted by e^{-I_E}, where the Euclidean action (including the FP-ghost terms) is

$$I_E = T \int d^2z \left\{ \partial X \cdot \bar{\partial} X + \frac{i}{T} \left(b\bar{\partial}c + \tilde{b}\partial\tilde{c} \right) \right\}, \qquad \left(\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} \right). \tag{8.3}$$

Now we can use conformal invariance to map the strip (8.2) in the complex z-plane to the entire complex w-plane, except the origin, by choosing

$$w = e^{-iz}. \tag{8.4}$$

Circles of constant $|w|$ now represent the string at fixed τ. The incoming string has been mapped to the origin of the w-plane (south pole of the Riemann sphere) and the outgoing string has been mapped to the point at infinity (north pole of the Riemann sphere). Because the action I_E of (8.3) is conformally invariant, the conformal mapping of (8.4) has no effect. The FP ghosts are crucial to this because for any CFT with central charge c,

$$L_0|_{cylinder} = L_0|_{R.sphere} + \frac{c}{24}. \tag{8.5}$$

The central charge therefore is therefore the Casimir energy due to the periodic boundary conditions; it is zero for the NG string in conformal gauge once account is taken of the FP ghosts.

The poles of the Riemann sphere are obviously special points; the Euclidean worldsheet is really the Riemann sphere with punctures at the poles; at these punctures we must provide information about the state of the string. This can be done by making an appropriate "insertion" in the path integral; there must be one possible insertion for each particle in the string spectrum. Leaving aside, for the moment, the precise nature of these "insertions", there is now an obvious way to generalise the path-integral to scattering amplitudes for particles in the string spectrum. We just consider more punctures of the Riemann sphere, each with its own insertion corresponding to a chosen state of the string; each puncture then represents a string in a particular state either incoming from the far past or outgoing in the far future[26]. For 4 punctures this will give us the scattering amplitude for the collision of two particles.

There is a further obvious generalisation in which the Riemann sphere is replaced by some other Riemann surface. In fact, a sum over all possibilities is obligatory, so the full amplitude will involve a sum over all relevant (closed, oriented) Riemann surfaces, which are classified by their genus g (number of handles). For example, the Riemann sphere has $g = 0$ and the torus has $g = 1$. This sum over g is the analog of the QFT loop-expansion, which is an expansion in powers of $\hbar$, so the contribution from the Riemann sphere is the analog of the "classical" term (given by tree Feynman diagrams in QFT). We shall focus on this zeroth-order contribution to the amplitude.

The restriction to $g = 0$ will allow us to now ignore the FP ghosts. Another simplification is that all Euclidean metrics on the Riemann-sphere are gauge-equivalent to the standard flat metric, so there will be no sum over Riemann spheres. Contrast this with the particle case, where we could use gauge invariance to set $e = s$ but we could not use it to set s to a particular value, so we were left with an integral over s. A theorem of Riemann assures us that there are no analogous "modular parameters" for the Riemann sphere, but we will have to sum over the positions of the punctures on it. Allowing for N punctures, we now have a path-integral of the form

$$\int [DX] e^{-I_E[X]} \prod_{i=1}^{N} \int d^2 z_i V_i(z, \bar{z}), \qquad (8.6)$$

where the quantities V_i are "vertex operators", one for each puncture at $z = z_i$, chosen from a set of possibilities in 1-1 correspondence with physical string states. They are not actually operators in the context of path-integral quantization, but they become operators upon canonical quantization, of conformal dimension $(1, 1)$ for a particular operator ordering. Any operator representing a physical state of the string must have conformal dimension $(1, 1)$ because this is what the constraints $(L_0 - 1)|\text{phys}\rangle = 0 = (\tilde{L}_0 - 1)|\text{phys}\rangle$ tell us.

[26]We will get the amplitude for this process in Euclidean space, and we then have to analytically continue to get the amplitude for Minkowski spacetime.

One way to deduce what we must choose for the V_i is to consider a modification of the string action to allow for interaction with background fields. For example, suppose that we have a spacetime with metric $g_{mn} = \eta_{mn} + h_{mn}(X)$. Then

$$I_E \to I'_E = I_E + T \int d^2 z \, \partial X^m \bar\partial X^n h_{mn}(X) \,.$$

A problem with this modification of the action is that it introduces interactions into what was a free 2D field theory. If we assume that h_{mn} is a small perturbation of the Minkowski metric then conformal invariance of the quantum theory restricts h_{mn} to be a physical solution of the linearised Einstein field equations. For example[27],

$$h_{mn}(X) = e^{i(p/2)\cdot X}\epsilon_{mn} \,, \qquad p^2 = 0 \,, \quad p^m \epsilon_{mn} = 0 \,. \tag{8.7}$$

For this choice, the modification of the NG conformal gauge action is consistent, and we now have a path integral weighted by

$$e^{-I'_E} = e^{-I_E} \left[1 + T \int d^2 z \, \partial X^m \bar\partial X^n h_{mn}(X) + \mathcal{O}(h^2) \right] \,. \tag{8.8}$$

This allows us to identify the graviton vertex "operator" as

$$V_{\text{graviton}} = e^{i(p/2)\cdot X}\epsilon_{mn}\partial X^m \bar\partial X^n \,. \tag{8.9}$$

Classically, the $\partial X \bar\partial X$ factor has conformal dimension $(1,1)$ and this remains true in the quantum theory provided $p^m \epsilon_{mn} = 0$.

The factor $e^{i(p/2)\cdot X}$ has no obvious conformal dimension but in the quantum theory one finds, for an appropriate operator ordering and after a long calculation, that, for the closed string,

$$\left[e^{i(p/2)\cdot X} \right] = \left(\frac{p^2}{8\pi T}, \frac{p^2}{8\pi T} \right) \,. \tag{8.10}$$

This shows that $e^{i(p/2)\cdot X}$ has conformal dimension $(1,1)$ when $p^2 = 8\pi T$, so it is the closed string tachyon vertex operator; for the open string one omits the factor of $1/2$ in the exponent to get an operator of conformal dimension 1 when $p^2 = 2\pi T$.

We need $p^2 = 0$ in the graviton vertex operator so that the $e^{i(p/2)\cdot X}$ factor does not move the conformal dimension away from $(1,1)$. At the next level there is a spin-3 particle with vertex operatorr

$$e^{i(p/2)\cdot X}\epsilon_{mnp}\partial X^m \partial X^n \bar\partial^2 X^p \qquad p^m \epsilon_{mnp} = 0 \,, \quad p^2 = -8\pi T \,. \tag{8.11}$$

As $(\partial X)^2 \bar\partial^2 X$ has conformal dimension $(2,2)$ for physical polarizations we now need $e^{i(p/2)\cdot X}$ to have conformal dimension $(-1,-1)$, hence $p^2 = -8\pi T$. Proceeding in this way, the physical states at every level of the string spectrum can be associated with a vertex operator of conformal dimension $(1,1)$.

---

[27]The factor of $1/2$ in the exponent will be explained below.

8.1 Virasoro-Shapiro amplitude

Let's consider the amplitude found by inserting the tachyon vertex operator at N points on the Riemann sphere:

$$A(p_1, \ldots, p_N) = \int [dX] \, e^{-I_E} \prod_{i=1}^{N} \int d^2 z_i \, e^{i(p/2)_i \cdot X_i} \,. \tag{8.12}$$

We can rewrite this as

$$A(p_1, \ldots, p_N) = \int [dX] \prod_{i=1}^{N} \int d^2 z_i \, e^{-I_E + \frac{i}{2} \sum_{j=1}^{N} p_j \cdot X_j} \,. \tag{8.13}$$

Now we observe that

$$-I_E + \frac{i}{2} \sum_{j=1}^{N} p_j \cdot X_j = -T \int d^2 z \left\{ \partial X \cdot \bar{\partial} X - \frac{i}{2T} \left[\sum_{j=1}^{N} \delta^2(z - z_j) p_j \right] \cdot X \right\}. \tag{8.14}$$

Integrating by parts, we can replace $\partial X \cdot \bar{\partial} X = -X \cdot \nabla^2 X$ since ∂ is acting on functions defined on the Riemann sphere, which has no boundary. This gives us

$$-I_E + \frac{i}{2} \sum_{j=1}^{N} p_j \cdot X_j = T \int d^2 z \left\{ X \cdot \left[\nabla^2 X + \frac{i}{2T} \sum_{j=1}^{N} \delta^2(z - z_j) p_j \right] \right\}. \tag{8.15}$$

The idea now is to complete the square in $X(z)$ but to do this we need to invert ∇^2 and there is a problem with this because ∇^2 has a zero eigenvalue on the sphere. The eigenfunction is the constant function, i.e. $X(z) = X_0$, so we should write

$$\int [dX] = \int d^D X_0 \int [dX]', \tag{8.16}$$

where $[dX]'$ is an integral over all functions *except* the constant function. Isolating the X_0-dependence we now have

$$A(p_1, \ldots, p_N) = \left[\int d^D X_0 \, e^{i(\sum_j p_j) \cdot X_0} \right] \hat{A}(p_1, \ldots, p_N)$$

$$\propto \delta \left(\sum_{j=1}^{N} p_j \right) \hat{A}(p_1, \ldots, p_N), \tag{8.17}$$

where the path integral for $\hat{A}$ excludes the integration over the constant function. The delta-function prefactor imposes conservation of the total D-momentum.

We can now invert ∇^2; the inverse is the 2D Green function:

$$\nabla^2 G(z, z_i) = \delta^2(z - z_i) \quad \Rightarrow \quad G(z, z_i) = \frac{1}{2\pi} \ln |z - z_i|. \tag{8.18}$$

Setting

$$X(z) = Y(z) - \frac{i}{4T} \sum_{i=1}^{N} G(z, z_i) p_i , \qquad (8.19)$$

we have[28] $[dX]' = [dY]'$, and [Exercise]

$$-I_E + \frac{i}{2} \sum_{j=1}^{N} p_j \cdot X_j = T \int d^2z \ Y \cdot \nabla^2 Y \ + \ \frac{1}{4\pi T} \sum_i \sum_j p_i \cdot p_j \ln |z_i - z_j| . \quad (8.20)$$

The terms in the double sum are infinite when $i = j$, but also independent of the momenta, so these terms can be omitted; they can only affect the overall normalisation. We can now do the Gaussian $[dY]'$ path integral, which also contributes only to the overall normalisation. We are then left with

$$\hat{A}(p_1, \ldots, p_N) \ \propto \ \prod_{i=1}^{N} \int d^2z_i \prod_{j<k} |z_j - z_k|^{\alpha_{jk}} , \qquad \alpha_{ij} = \frac{p_i \cdot p_j}{2\pi T} . \qquad (8.21)$$

As the derivation of this formula assumed conformal invariance, this result should be invariant under the $Sl(2; \mathbb{C})$ conformal isometry group of the Riemann sphere. The $Sl(2; \mathbb{C})$ transformation of z is

$$z \to z' = \frac{az + b}{cz + d} , \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 . \qquad (8.22)$$

Using this one finds that

$$z_i' - z_j' = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)} , \qquad d^2z' = \frac{dz^2}{(cz + d)^4} , \qquad (8.23)$$

and hence that

$$\prod_{i=1}^{N} d^2z_i' \prod_{j<k} |z_j' - z_k'|^{\alpha_{jk}} = \left[\prod_{i=1}^{N} d^2z_i \prod_{j<k} |z_j - z_k|^{\alpha_{jk}} \right] \left[\prod_{i=1}^{N} |cz_i + d|^{-4 - \sum_j' \alpha_{ij}} \right] , \quad (8.24)$$

where

$$\sum_j {}' \alpha_{ij} = \sum_{j=1}^{N} \alpha_{ij} - \alpha_{ii} \qquad (i = 1, \ldots, N)$$

$$= \frac{1}{2\pi T} p_i \cdot \left(\sum_{j=1}^{N} p_j \right) - \frac{p_i^2}{2\pi T}$$

$$= -\frac{p_i^2}{2\pi T} \qquad \text{(by momentum conservation).} \qquad (8.25)$$

---

[28]A shift in the integration variable has no effect because we integrate over all values of the (non-constant) functions X.

We see from this that the amplitude is $Sl(2;\mathbb{C})$ invariant only if

$$-4 + \frac{p_i^2}{2\pi T} = 0 \quad \Leftrightarrow \quad p_i^2 = 8\pi T \,. \tag{8.26}$$

This is the mass-shell condition for the tachyonic ground state of the string!

The $Sl(2;\mathbb{C})$ isometry group of the Riemann sphere implies that the positions of any three punctures can be chosen arbitrarily, so the expression (8.21) has three too many integrals. We could fix this problem by the gauge choice[29]

$$f_i \equiv z_i - u_i = 0 \quad i = 1, 2, 3. \tag{8.27}$$

Inserting $\delta^2(f_1)\delta^2(f_2)\delta^2(f_3)$ into the integrand of (8.21) removes three of the integrals, but this would not be the correct thing to do. The problem is that the $Sl(2;\mathbb{C})$ invariance of the amplitude implies that it is proportional to the volume of $Sl(2;\mathbb{C})$, which is infinite because the group is non-compact, but this factor does not appear *explicitly* in the expression (8.21) for the amplitude.

We know how to solve this problem. When we fix the positions of the first three points by insertion of delta functions, we must also include a Fadeev-Popov determinant. The infinitesimal form of the transformation (8.22) is[30]

$$\delta z = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \,. \tag{8.28}$$

From this we compute

$$\left\| \frac{\partial f_i}{\partial \alpha_j} \right\| = \left\| \begin{matrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{matrix} \right\| = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1) \,. \tag{8.29}$$

Because the variables are complex (we are inserting three 2D delta functions), the FP determinant is the modulus squared of this, so

$$\Delta_{FP} = |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2 \,. \tag{8.30}$$

Following the earlier argument for gauge fixing the particle action, the insertion of the delta functions with the FP determinant allows us to factor out the (infinite) volume Ω of $Sl(2;\mathbb{C})$; dividing by this volume we then get

$$\Omega^{-1} \hat{A}(p_1, \ldots, p_N) \; \propto \; \prod_{i=1}^{N} \int d^2 z_i \, \delta^2(f_1) \delta^2(f_2) \delta^2(f_3) \Delta_{FP} \prod_{j<k} |z_j - z_k|^{\alpha_{jk}} \,. \tag{8.31}$$

---

[29] Global isometries correspond to zero modes of FP ghost c, so we must omit the integration over these modes in the path integral. This means that global isometries remain as gauge invariances that we have to deal with at this stage.

[30] This shows that $(\partial, z\partial, z^2\partial)$ are the globally defined conformal Killing vector fields.

This can be checked as follows. Multiply both sides by $|(u_1 - u_2)(u_2 - u_3)(u_3 - u_1)|^{-2}$ and integrate over (u_1, u_2, u_3). On the RHS the u integrals can be done using the delta functions, the Δ_{FP} factor is then cancelled and we recover the expression (8.21). On the LHS the integral cancels the factor of Ω^{-1} because, formally,

$$\Omega = \int \frac{d^2 u_1 d^2 u_2 d^2 u_3}{|(u_1 - u_2)(u_2 - u_3)^2(u_3 - u_1)|^2} . \tag{8.32}$$

This integral is infinite but the integrand is the $Sl(2; \mathbb{C})$ invariant measure on the $Sl(2; \mathbb{C})$ group manifold, parametrised by three complex coordinates on which $Sl(2; \mathbb{C})$ acts by the fractional linear transformation (8.22).

We may now do the (z_1, z_2, z_3) integrals of (8.31) to get

$$\Omega^{-1} \hat{A}(p_1, \ldots, p_N) \propto |u_1 - u_2|^{2+\alpha_{12}} |u_2 - u_3|^{2+\alpha_{23}} |u_3 - u_1|^{2+\alpha_{13}} \tag{8.33}$$

$$\times \prod_{i=4}^{N} \int d^2 z_i \prod_{i=4}^{N} |u_1 - z_i|^{\alpha_{1i}} |u_2 - z_i|^{\alpha_{2i}} |u_3 - z_i|^{\alpha_{3i}} \prod_{4 \leq j < k} |z_i - z_j|^{\alpha_{jk}} .$$

This can be simplified enormously by the choice

$$u_3 = 1 , \qquad u_2 = 0 , \qquad u_1 \to \infty . \tag{8.34}$$

In this limit we get a factor of

$$|u_1|^{4 - \alpha_{11}^2 + \sum_i \alpha_{1i}} = 1 , \tag{8.35}$$

where the equality follows upon using both the mass-shell condition and momentum conservation. The remaining terms give the Virasoro-Shapiro amplitude

$$\hat{A}_{VS}(p_1, \ldots, p_N) = \prod_{i=4}^{N} \int d^2 z_i \prod_{i=4}^{N} |z_i|^{\alpha_{2i}} |z_i - 1|^{\alpha_{3i}} \prod_{4 \leq j < k} |z_i - z_j|^{\alpha_{jk}} . \tag{8.36}$$

The result for $N = 3$ is a constant, which can be interpreted as a coupling constant. For $N = 4$ we have the Virasoro amplitude

$$\hat{A}(p_1, p_2, p_3, p_4) = \int d^2 z \, |z|^{\alpha_{24}} |z - 1|^{\alpha_{34}} . \tag{8.37}$$

8.1.1 The Virasoro amplitude

Consider the elastic scattering of two identical particles of mass m. In the rest frame, the incoming particles have D-momenta

$$p_1 = (E, \vec{p}) , \qquad p_2 = (E, -\vec{p}) . \tag{8.38}$$

The outgoing particles have D-momenta

$$-p_3 = (E, \vec{p}') , \qquad -p_4 = (E, -\vec{p}') . \tag{8.39}$$

Notice that $p_1 + p_2 + p_3 + p_4 = 0$, as required by D-momentum conservation. In addition, since $p_i^2 = -m^2$ for $i = 1, 2, 3, 4$, we have

$$|\vec{p}|^2 = |\vec{p}'|^2 = E^2 - m^2 \,. \tag{8.40}$$

The scattering angle θ_s is defined by

$$\cos\theta_s = \frac{\vec{p} \cdot \vec{p}'}{E^2 - m^2} \,. \tag{8.41}$$

We may trade the frame-dependent variables $(E, \cos\theta_s)$ for the Lorentz scalar Mandelstam variables

$$
\begin{aligned}
s_M &= -(p_1 + p_2)^2 = 4E^2 \\
t_M &= -(p_1 + p_3)^2 = -2(E^2 - m^2)(1 - \cos\theta_s) \\
u_M &= -(p_1 + p_4)^2 = -2(E^2 - m^2)(1 + \cos\theta_s)
\end{aligned}
\tag{8.42}
$$

These are not all independent because

$$s_M + t_M + u_M = 4m^2 \,. \tag{8.43}$$

The variable s_M is the square of the centre of mass energy. For fixed s_M, the variable t_M determines the scattering angle.

In the context of the closed string it is convenient to use the rescaled Mandelstam variables[31]

$$(s, t, u) = \frac{1}{8\pi T}(s_M, t_M, u_M) \,. \tag{8.44}$$

For tachyon scattering, we have

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 8\pi T \,, \tag{8.45}$$

and hence

$$
\begin{aligned}
s &= -\frac{1}{8\pi T}(p_1 + p_2)^2 = -2 - \frac{1}{2}\alpha_{12} \\
t &= -\frac{1}{8\pi T}(p_1 + p_3)^2 = -2 - \frac{1}{2}\alpha_{13} \\
u &= -\frac{1}{8\pi T}(p_1 + p_4)^2 = -2 - \frac{1}{2}\alpha_{14}
\end{aligned}
\tag{8.46}
$$

From (8.42) and momentum conservation it follows that

$$
\begin{aligned}
\alpha_{34} &= \alpha_{12} = -4 - 2s \,, \\
\alpha_{24} &= \alpha_{13} = -4 - 2t \,, \\
\alpha_{23} &= \alpha_{14} = -4 - 2u \,,
\end{aligned}
\tag{8.47}
$$

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[31]Equivalently, we can choose units for which $8\pi T = 1$.

and
$$s + t + u = -4 \,. \tag{8.48}$$

We can also write s and t as

$$s = \frac{E^2}{2\pi T} \,, \quad t = -2\left(1 + \frac{s}{4}\right)(1 - \cos\theta_s) \,. \tag{8.49}$$

Using (8.47) we can rewrite the $N = 4$ amplitude of (8.37) as

$$\hat{A}(p_1, p_2, p_3, p_4) = \int d^2 z \, |z|^{2\alpha} |z - 1|^{2\beta} \,, \quad \alpha = -2 - t \,, \quad \beta = -2 - s \,. \tag{8.50}$$

Next, using the identity

$$\frac{1}{\pi} \int d^2 z \, |z|^{2\alpha} |z - 1|^{2\beta} \equiv \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)\Gamma(-\alpha - \beta - 1)}{\Gamma(\alpha + \beta + 2)\Gamma(-\alpha)\Gamma(-\beta)} \,, \tag{8.51}$$

we arrive at the *Virasoro amplitude* for the scattering of two closed string tachyons:

$$A(s, t) \; \propto \; \frac{\Gamma(-1 - t)\Gamma(-1 - s)\Gamma(-1 - u)}{\Gamma(u + 2)\Gamma(s + 2)\Gamma(t + 2)} \quad (u = -4 - s - t). \tag{8.52}$$

Here, $\Gamma(z)$ is Euler's Gamma function, with the properties

$$z\Gamma(z) = \Gamma(z + 1) \,, \quad \Gamma(n) = (n - 1)! \quad \text{for } n \in \mathbb{Z}^+ \,. \tag{8.53}$$

It also has an analytic continuation to a meromorphic function on the complex z-plane with simple poles at $z = -n$, with residues $(-1)^n/n!$, for $n \geq 0$.

For fixed generic t the Virasoro amplitude A becomes a function of s with simple poles at

$$s = -1, 0, 1, 2, \ldots \tag{8.54}$$

These poles correspond to resonances, i.e. to other particles in the spectrum (stable particles, in fact, because the poles are on the real axis in the complex s-plane). The position of the pole on the real axis gives the mass-squared of the particle in units of $8\pi T$. The pole at $s = -1$ is the tachyon itself; in other words, the tachyon can be considered as a bound state of two other tachyons. The pole at $s = 0$ implies the existence of a massless particle, or particles. The residue of this pole is

$$-\frac{\Gamma(-1 - t)\Gamma(3 + t)}{\Gamma(-2 - t)\Gamma(t + 2)} = t^2 - 4 \,. \tag{8.55}$$

This is a quadratic function of t and hence of $\cos\theta_s$, which implies that there must be a massless particle of spin 2 (but none of higher spin). The residue of the pole at $s = n$ is a polynomial in t of order $2(n + 1)$, so that $2(n + 1)$ is the maximum spin of particles in the spectrum with mass-squared $n \times (8\pi T)$. In a plot of J_{max} against s, such particles appear at integer values of J_{max} on a straight line with slope

$\alpha'/2$ and intercept 2 (value of J_{max} at $s = 0$). This is the leading Regge trajectory. All other particles in the spectrum appear on parallel "daughter" trajectories in the (J, s) plane (e.g. the massless spin-zero particle in the spectrum is the first one on the trajectory with zero intercept. In fact, the entire string spectrum can be found in this way!

If we had computed the amplitude for scattering gravitons instead of tachyons then we would have found the tachyon as a resonance. This shows that it is not consistent to simply omit the tachyon from the spectrum.

Another feature of the Virasoro amplitude is its $s \leftrightarrow t$ symmetry. Poles in A as a function of s at fixed t therefore reappear as poles in A as a function of t at fixed s. These correspond to the exchange of a particle. In particular, a massless spin-2 particle is exchanged, and general arguments imply that such a particle must be the quantum associated to the gravitational force, so a theory of interacting closed strings is a theory of quantum gravity.

The Virasoro amplitude for closed strings was preceded by the Veneziano amplitude for open strings[32]

$$A(s, t) = \frac{\Gamma(-1 - s)\Gamma(-1 - t)}{\Gamma(-2 - s - t)}, \qquad (8.56)$$

where now

$$s = -\frac{1}{2\pi T} (p_1 + p_2)^2, \qquad t = -\frac{1}{2\pi T} (p_1 + p_3)^2. \qquad (8.57)$$

This amplitude also has poles at $s = -1, 0, 1, 2, \ldots$, but the maximum spin for $s = n$ is now $J_{\max} = n + 1$, and the leading Regge trajectory has slope α' and intercept 1 (this is the constant a that equals the zero point energy in the light-cone gauge quantization of the open string).

8.2 String theory at 1-loop: taming UV divergences

We will now take a brief look at what happens at one string-loop. In this case amplitudes are found from the path integral by considering vertex operator insertions at points on a conformally flat complex torus. We can define a flat torus by a doubly periodic identification in the complex z-plane. Without loss of generality (because we are free to rescale z) we can choose

$$z \sim z + 1, \qquad z \sim z + \tau \qquad \mathcal{I}m\,\tau > 0. \qquad (8.58)$$

Any further analytic transformation that preserves the first of these identifications will leave τ unchanged[33]. However, not all values of τ in the upper half plane define

---

[32]Veneziano did not compute it from string theory (which did not then exist) but just proposed it on the basis of its properties.

[33]By a *non-analytic* coordinate transformation we can put the equivalence relations into the form $z \sim z + n + im$ for integers (n, m), but the metric is then conformal to $|dz + \mu d\bar{z}|^2$, and μ now parametrises the conformally inequivalent metrics on the torus with standard identifications.

inequivalent tori because the identifications are obviously unchanged by the translation

$$T: \quad \tau \to \tau + 1 \,. \tag{8.59}$$

They are also unchanged by the inversion

$$S: \quad \tau \to -\frac{1}{\tau} \,. \tag{8.60}$$

To see this multiply the equivalence relations of (8.58) by $-1/\tau$ and then rescale $z \to -\tau z$ to get[34]

$$z \sim z - \frac{1}{\tau} \,, \qquad z \sim z - 1 \,. \tag{8.61}$$

Composition of the S and T maps generates elements of the group $PSl(2; \mathbb{Z})$ which acts on τ by a fractional linear transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2; \mathbb{Z}) \tag{8.62}$$

Two matrices of $Sl(2; \mathbb{Z})$ that differ by a sign have the same action on τ, so $PSl(2; \mathbb{Z}) \cong Sl(2; \mathbb{Z})/\{\pm 1\}$. Inequivalent tori are parametrised by complex numbers τ lying in a fundamental domain of $PSl(2; \mathbb{Z})$ in the complex τ-plane.

In the path integral representation of the one string-loop amplitudes, we have to sum over all inequivalent tori (Euclidean worldsheets of genus 1). This leads to an integral over τ:

$$A \propto \int_F d^2\tau \dots \tag{8.63}$$

where F is any fundamental domain of $PSl(2; \mathbb{Z})$; it is convenient to choose it to be the one in which we may take $\mathcal{I}m\,\tau \to \infty$ on the imaginary axis. In this limit the torus becomes long and thin and it starts to look like a one-loop Feynman diagram (with vertices at various points if we had vertex operators at points on the torus). In this *infra-red* limit we can interpret $\mathcal{I}m\,\tau$ as the modular parameter s of a particle worldline. In the particle case we would have

$$A \propto \int_0^\infty ds \dots \tag{8.64}$$

and we typically get divergent results from the part of the integral where $s \to 0$. These are *ultra-violet* divergences. They are absent in string theory because the domain of integration F does not include the origin of the τ-plane.

Moral There are no UV divergences in string theory. We have discussed only one string-loop but the result is general. This can be understood in other ways. For

---

[34]The relation $z \sim z + 1$ is equivalent to $z \sim z - 1$.

example, the force due to exchange of the massless states of the closed string includes the force of gravity because it includes a spin-2 field. Whereas this leads to unacceptable UV behaviour in GR, the UV divergences are cut off in string theory at the string length scale

$$\ell_s \sim \sqrt{\alpha'} \tag{8.65}$$

because at this scale the exchange of the massive string states becomes as important as the graviton exchange.

This led to the idea (not entirely correct, as we shall see) that we should identify the string length with the Planck length set by Newtons's constant.

8.3 The dilaton and the string-loop expansion[*]

The way that the dilaton field $\Phi(X)$ couples to the string in its Polyakov formulation is through the scalar curvature of the independent worldsheet metric $\gamma_{\mu\nu}$. In two dimensions the Riemann curvature tensor is entirely determined by its double trace, the Ricci scalar $R(\gamma)$, but this allows us to add to the Euclidean NG action the term

$$I_\Phi = \frac{1}{4\pi} \int d^2z \, \Phi(X)\sqrt{\gamma} \, R(\gamma) \,. \tag{8.66}$$

Here are some features of this term:

- If $\Phi = \phi_0$, a constant then

$$I_\Phi = \phi_0 \, \chi \,, \qquad \chi = \frac{1}{4\pi} \int d^2z \, \sqrt{\gamma} \, R(\gamma) \,. \tag{8.67}$$

  The integral χ is a topological invariant of the worldsheet, called the Euler number. For a compact orientable Riemann surface without boundary (which we'll abbreviate to "Riemann surface" in what follows) the Euler number is related to the genus g (the number of doughnut-type "holes") by the formula

$$\chi = 2\,(1 - g) \,. \tag{8.68}$$

- In conformal gauge, we can write the line element for the (Euclidean signature) metric γ as $ds^2(\gamma) = 2e^\sigma dz d\bar{z}$, i.e. a conformal factor e^σ (an arbitrary function of z and $\bar{z}$) times the Euclidean metric. We then find that $\sqrt{\gamma} \, R(\gamma) = 2\nabla^2\sigma$ and hence, after integrating by parts,

$$I_\Phi = \frac{1}{2\pi} \int d^2z \, \sigma \, \partial X^m \bar{\partial} X^n \partial_n \partial_m \Phi \,. \tag{8.69}$$

  This dependence on σ shows that I_Φ is *not* conformal invariant, unless Φ is constant. This is allowed because I_Φ, being independent of the string tension T, comes with an additional factor of α' relative to the NG action (we have to consider the lack of conformal invariance of I_Φ at the same time that we consider possible conformal anomalies).

These properties suggest that write

$$\Phi = \phi_0 + \phi(X), \tag{8.70}$$

where $\phi(X)$ is zero in the vacuum; i.e. the constant ϕ_0 is the "vacuum expectation value" of $\Phi(X)$. Then there will appear a factor in the path integral of the form

$$e^{-\phi_0 \chi} = \left(g_s^2\right)^{g-1}, \qquad g_s \equiv e^{\phi_0}. \tag{8.71}$$

For $g = 0$ this tells us that the Riemann sphere contribution to scattering amplitudes is weighted by a factor of $1/g_s^2$. If we use these amplitudes to construct an effective field theory action S from which we could read off the amplitudes directly (by looking at the various interaction terms) then this action will come with a factor of $1/g_s^2$ (we can then absorb all other dimensionless factors into a redefinition of g_s, i.e. of ϕ_0). If we focus on the amplitudes for scattering of massless particles then we find that

$$S[g, b, \phi] = \frac{1}{g_s^2} \frac{1}{\ell_s^{(D-2)}} \int d^D x \sqrt{-\det g} \, e^{-2\phi} \left[2\Lambda + R(g) - \frac{1}{3} H^2 + 4 \left(\partial \phi \right)^2 + \mathcal{O}(\alpha') \right], \tag{8.72}$$

where $H = db$ (field strength of the antisymmetric tensor field) and the cosmological constant is

$$\Lambda = \frac{(D - 26)}{3\alpha'}. \tag{8.73}$$

Some other features of the effective space-time action are

- The exact result for S will involve a series of all order in α' since the coupling of the background fields to the string introduces interactions into the 2D QFT defined by the string worldsheet action.

- The leading term is the cosmological constant Λ. Unless we know the entire infinite series in α', we must set $\Lambda = 0$; i.e. we must choose $D = 26$. It is then consistent to consider the string as a perturbation about the Minkowski vacuum, which is what we implicitly assumed when we earlier derived the condition $D = 26$.

- It is consistent to exclude the coupling to the string of the fields associated to massive modes of the string because without them the worldsheet action defines a renormalizable 2D QFT. Coupling to the fields associated to the massive particles in the string spectrum leads to a non-renormalizable 2D QFT for which it is necessary to consider all possible terms of all dimensions. But if all fields of level $N > 1$ are all zero initially then they stay zero.

- The integrand involves a factor of $e^{-2\phi}$. This is because the action must be such that $\phi_0 \equiv \ln g_s$ and $\phi(X)$ must appear only through the combination $\Phi = \phi_0 + \phi(X)$.

In the effective spacetime action, g_s^2 plays the role of $\hbar$. This suggests that we have been considering so far only the leading term in a semi-classical expansion. This is true because we have still to consider Riemann surfaces with genus $g > 0$, and a string amplitude at genus g is weighted, according to to (8.71), by a factor of $(g_s^2)^{g-1}$, i.e. a factor of $(g_s^2)^g$ relative to the zero-loop amplitude. This confirms that the string-loop expansion is a semi-classical expansion in powers of g_s^2. Taking into account all string loops gives us a *double expansion* of the effective field theory[35]

$$S = \frac{1}{g_s^2 \ell_s^{(D-2)}} \int d^D x \, \sqrt{-\det g} \sum_{g=0}^{\infty} g_s^{2g} \, e^{2(g-1)\phi} \, L_g \,, \qquad L_g = \sum_{l=0}^{\infty} \ell_s^{2l} L_g^{(l)} \,. \qquad (8.74)$$

In effect, the expansion in powers of ℓ_s comes from first-quantisation of the string, and the expansion in powers of g_s comes from second-quantisation. How can we quantise twice? Is there not a single $\hbar$? The situation is actually not so different from that of the point particle. When we first-quantise we get a Klein-Gordon equation but with a mass $m/\hbar$; we then relabel this as m so that it becomes the mass parameter of the classical field equation, and then we quantise again. For the string, first quantization would have led to $\alpha'\hbar$ as the expansion parameter if we had not set $\hbar = 1$; if we relabel this as α' then $\hbar$ appears only in the combination $g_s^2 \hbar$.

To lowest order in α' we have what looks like GR coupled to an antisymmetric tensor and a scalar. The D-dimensional Newton gravitational constant G_D is

$$G_D \; \propto \; g_s^2 \ell_s^{(D-2)} \,. \qquad (8.75)$$

Consistency of the string-loop expansion (in powers of g_s^2) relies on this formula. Particles in the string spectrum have masses proportional to $1/\ell_s$, independent of g_s, so their contribution to the gravitational potential in D dimensions is proportional to g_s^2, and hence zero at zero string coupling. This means that the strings of free ($g_s = 0$) string theory do not back-react on the space-time metric; the metric is changed by the presence of strings only within perturbation theory. If this had not been the case it would not have been consistent to start (as we did) by considering a string in Minkowski spacetime.

Why is g_s called the string coupling constant? Consider a g string-loop vacuum to vacuum diagram with the appearance of a chain of g tori connected by long "throats", and think of it as "fattened" Feymann diagram in which a chain of g loops connected by lines; where each line meets a loop we have a 3-point vertex. As

---

[35]There is a lot of freedom in the form of the Lagrangians $L_g^{(l)}$ beyond leading order. Recall that the construction of S involves a prior determination of scattering amplitudes of the level-1 fields, which we then arrange to replicate from a local spacetime Lagrangian. Since the amplitudes are all "on-shell" they actually determine only the field equations for the background fields, and then only up to field redefinitions. Even with all this freedom it is not obvious why it should be possible to replicate the string theory scattering amplitudes in this way, although this has been checked to low orders in the expansion and there are general arguments that purport to prove it.

there are $(g-1)$ lines that link the loops, and each of the two ends of each line ends at a vertex, we have a total of $2(g-1)$ vertices. If we associate a coupling constant to each vertex, call it g_s, we see that this particular diagram comes with a factor of $(g_s^2)^{g-1}$, which agrees with our earlier result.

Is there a g-loop Riemann surface with the appearance just postulated. Yes, there is. For $g > 0$ there is no longer a unique flat metric, as we have already seen for $g = 1$. For $g \geq 2$ there is a $3(g-1)$-parameter family of conformally inequivalent flat metrics; these parameters are called "moduli". This number can be understood intuitively from the "chain of tori" diagram if we associate one parameter with each propagator. For g loops we have, in addition to the $(g-1)$ links, $(g-2)$ "interior" loops with 2-propagators each, and two "end of chain" loops with one propagator each. The total number of propagators is therefore

$$(g-1) + 2(g-2) + 2 = 3(g-1). \tag{8.76}$$

This is also, and not coincidentally, the dimension of the space of quadratic differentials on a Riemann surface of genus $g \geq 2$.

9. Interlude: The spinning particle

A point particle with non-zero spin can be accommodated by including additional anticommuting coordinates. Consider the non-relativistic particle action

$$I = \int dt \left\{ \dot{\vec{x}} \cdot \vec{p} + \frac{i}{2} \vec{\psi} \cdot \dot{\vec{\psi}} - \frac{|\vec{p}|^2}{2m} \right\}, \tag{9.1}$$

where the 3-vector variables $\vec{\psi}$ are anti-commuting and "real". The Poisson brackets of the anticommuting variables are[36]

$$\left\{ \psi^a, \psi^b \right\}_{PB} = -i\delta^{ab}. \tag{9.2}$$

Upon quantization we get the canonical anticommutation relations $\left\{ \psi^a, \psi^b \right\} = \delta^{ab}$, which can be realized in terms of Pauli matrices: $\vec{\psi} = \vec{\sigma}/\sqrt{2}$.

The action is invariant under space rotations; the infinitesimal transformations are

$$\delta_\omega \vec{x} = \vec{\omega} \times \vec{x}, \qquad \delta_\omega \vec{p} = \vec{\omega} \times \vec{p}, \qquad \delta_\omega \vec{\psi} = \vec{\omega} \times \vec{\psi}. \tag{9.3}$$

The Noether charge is the angular momentum $\vec{J} = \vec{L} + \vec{S}$. The first term is the standard position-dependent orbital angular momentum: $\vec{L} = \vec{x} \times \vec{p}$. The second

---

[36]If we had $i\psi^* \dot{\psi}$ for complex anticommuting ψ then the PB would be $\{\psi^*, \psi\}_{PB} = -i$, consistent with our earlier conventions for $ib\dot{c}$. Writing $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$ for "real" (ψ_1, ψ_2) then gives us $\frac{1}{2}\left[\psi_1 \dot{\psi}_1 + \psi_2 \dot{\psi}_2\right]$ with PBs $\{\psi_i, \psi_j\}_{PB} = -i\delta_{ij}$. This explains why the factor of $1/2$ is needed in (9.1) to get the PB as given.

term is the spin: $\vec{S} = -\frac{i}{2}\vec{\psi} \times \vec{\psi}$. Upon quantization, and using the properties of the Pauli matrices, we find that

$$\vec{S} = \frac{1}{2}\vec{\sigma}\,, \tag{9.4}$$

which shows that the action (9.1) describes a spin-$\frac{1}{2}$ particle.

The relativistic generalization, in D spacetime dimensions, is simplest for a massless particle[37]. The spin can be accommodated by including an anticommuting D-vector coordinate ψ^m. The action is

$$I = \int dt \left\{ \dot{x}^m p_m + \frac{i}{2}\eta_{mn}\,\psi^m \dot{\psi}^n - \frac{1}{2}e\,p^2 - i\chi\,\psi \cdot p \right\}\,. \tag{9.5}$$

We read off from this action the Poisson brackets

$$\{x^m, p_n\}_{PB} = \delta_n^m\,, \qquad \{\psi^m, \psi^n\}_{PB} = -i\eta^{mn}\,. \tag{9.6}$$

Using this we find that

$$\{\psi \cdot p, \psi \cdot p\}_{PB} = -ip^2\,, \qquad \{p^2, \psi \cdot p\}_{PB} = 0\,, \tag{9.7}$$

which shows that the constraints are first-class and hence that they both generate gauge invariances. These are

$$\delta x^m = \alpha p^m + i\epsilon\psi^m\,, \qquad \delta\psi^m = -\epsilon p^m\,, \tag{9.8}$$

where ϵ is an infinitesimal anticommuting parameter. The action is invariant if the Lagrange multipliers transform as

$$\delta e = \dot{\alpha}\,, \qquad \delta\chi = \dot{\epsilon}\,. \tag{9.9}$$

To be precise, one finds that

$$\delta I = \frac{1}{2}\left[\alpha\,p^2 + i\epsilon\,\psi \cdot p\right]_{t_A}^{t_B}\,, \tag{9.10}$$

which is zero if α and ϵ are zero at $t = t_A$ and $t = t_B$.

The simplest way to see that this action does indeed describe a massless spin-$\frac{1}{2}$ relativistic particle is to quantize *à la* Dirac. Recall that we first quantize as if there were no constraints, which gives us the (anti)commutation relations

$$[x^m, p_n] = i\delta_m^n\,, \qquad \{\psi^m, \psi^n\} = \eta^{mn}\,. \tag{9.11}$$

These can be realised on a spinor wave function $\Psi(x)$ by

$$p_m = -i\partial_m\,, \qquad \psi^m = \Gamma^m/\sqrt{2}\,, \tag{9.12}$$

---

[37]The massive case can be found by starting with the massless particle action (9.5) in $(D+1)$ dimensions and reducing to D dimensions with $P_D = m$.

where Γ^m are the $2^{[D]/2} \times 2^{[D]/2}$ Dirac matrices. Next, we impose the constraints as physical state conditions. We only have to impose the condition resulting from the constraint $\psi \cdot p = 0$ because this implies the mass-shell condition; the result is the massless Dirac equation

$$\Gamma^m \partial_m \Psi(x) = 0 \,. \tag{9.13}$$

If $D = 5, 6, 7$ mod 8 then Ψ must be complex, i.e. a "Dirac spinor", but if $D = 2, 3, 4, 8, 9$ mod 8 then we may choose it to be real[38]. For any D we can define

$$\Gamma_{D+1} = \Gamma^0 \Gamma^1 \cdots \Gamma^D \,, \tag{9.14}$$

but this matrix is ± 1 for odd D; for even D it anticommutes with each of the D Dirac matrices and its square is either $+1$ or -1. If $\Gamma_{D+1}^2 = 1$ we can define a chiral (anti-chiral) spinor as an eigenspinor of Γ_{D+1}. For $D = 2$ mod 8, but not otherwise, a real spinor can also be chiral. This fact is of importance for superstring theory.

10. The spinning string

The spinning string follows the example of the spinning particle. The action is a generalization of the NG action to include an anti-commuting Lorentz vector 2-component spinor worldsheet field ψ^m. Actually, the one-component chiral (antichiral) projections $\psi_\pm^m$ are separately representations of the 2D Lorentz group, so we have the option of including just one of the two components, either ψ_-^m or ψ_+^m. This leads ultimately to the heterotic strings, but here we stick to the simpler case in which both chiral components of ψ^m are included; for a closed string this leads to the Type II strings, and for an open string it leads to the Type I string.

Having introduced the anti-commuting variables $\psi_\pm^m$ we now need a gauge invariance to remove their unphysical time components. In the context of the Polyakov action, which is just worldsheet scalars X^m coupled to "2D gravity", what we need is a generalization to "2D supergravity". However, we shall go straight to the Hamiltonian form of the action, where the new gauge invariances are implemented by new constraints with anti-commuting constraint functions, which we shall call $\mathcal{Q}_\pm$. The new constraints are imposed in the action by new anti-commuting Lagrange multipliers $\chi^\pm$. The closed spinning string action therefore takes the form

$$I = \int dt \oint d\sigma \left\{ \dot{X}^m P_m + \frac{i}{2} T \left(\psi_+ \dot{\psi}_+ + \frac{i}{2} \psi_- \dot{\psi}_- \right) \right. $$
$$\left. - \lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+ - i\chi^- \mathcal{Q}_- - i\chi^+ \mathcal{Q}_+ \right\} \,. \tag{10.1}$$

---

[38]This can be interpreted as a "gauging" of the time-reversal invariance of the particle action. In general, time reversal is represented in QM by an anti-unitary operator K, with the property that $K^2 = \pm 1$. When $K^2 = 1$, as is the case for the spinning particle when $D = 2, 3, 4, 8, 9$ mod 8, we can impose the condition $K\Psi = \pm\Psi$ as a new physical state condition. This is a reality condition because K involves taking the complex conjugate. When $K^2 = -1$, as is the case for the spinning particle when $D = 5, 6, 7$ mod 8, it is not consistent to impose a reality condition so Ψ is necessarily complex; this is a simple illustration of Kramer's degeneracy in QM.

The constraint functions are

$$\mathcal{H}_\pm = \frac{1}{4T}\left(P \pm TX'\right)^2 \mp \frac{i}{2}T\psi_\pm \cdot \psi'_\pm \,, \qquad \mathcal{Q}_\pm = \frac{1}{2}\left(P \pm TX'\right)\cdot\psi_\pm \,. \tag{10.2}$$

This action reduces to the Hamiltonian form of the NG action when all anti-commuting variables are omitted. This means that all classical solutions of the NG string are solutions of the spinning string, but the new anti-commuting variables make a significant difference to the quantum theory.

We can read off the non-zero PB relations from the above action. In particular,

$$\left\{\psi_\pm^m(\sigma), \psi_\pm^n(\sigma')\right\}_{PB} = -i\eta^{mn}\delta(\sigma - \sigma')\,. \tag{10.3}$$

A calculation using the PBs of the canonical variables shows that the non-zero PBs of the constraint functions are

$$\left\{\mathcal{Q}_\pm(\sigma), \mathcal{Q}_\pm(\sigma')\right\}_{PB} = -i\mathcal{H}_\pm(\sigma)\delta(\sigma - \sigma')$$

$$\left\{\mathcal{Q}_\pm(\sigma), \mathcal{H}_\pm(\sigma')\right\}_{PB} = -\left[\frac{1}{2}\mathcal{Q}_\pm(\sigma) + \mathcal{Q}_\pm(\sigma')\right]\delta'(\sigma - \sigma')$$

$$\left\{\mathcal{H}_\pm(\sigma), \mathcal{H}_\pm(\sigma')\right\}_{PB} = \mp\left[\mathcal{H}_\pm(\sigma) + \mathcal{H}_\pm(\sigma')\right]\delta'(\sigma - \sigma')\,. \tag{10.4}$$

From this we see that the constraints are all first-class, so they generate gauge invariances of the canonical variables via their PB with $\xi^\pm\mathcal{H}_\pm + i\epsilon^\pm\mathcal{Q}_\pm$ for parameters $\xi^\pm$ and anticommuting parameters $\epsilon^\pm$. One then finds that the action is invariant provided the Lagrange multipliers are assigned the gauge transformations

$$\delta_\xi\lambda^\pm = \dot{\xi}^\pm \mp \lambda^\pm\left(\xi^\pm\right)' \pm \xi^\pm\left(\lambda^\pm\right)'\,, \qquad \delta_\xi\chi^\pm = \mp\frac{1}{2}\left(\xi^\pm\right)'\chi^\pm \pm \xi^\pm\left(\chi^\pm\right)'\,,$$

$$\delta_\epsilon\lambda^\pm = i\chi^\pm\epsilon^\pm\,, \qquad \delta_\epsilon\chi^\pm = \dot{\epsilon}^\pm \mp \lambda^\pm\left(\epsilon^\pm\right)' \pm \frac{1}{2}\left(\lambda^\pm\right)'\epsilon^\pm\,. \tag{10.5}$$

10.1 Conformal gauge and superconformal symmetry

The conformal gauge for the spinning string is

$$\lambda = \tilde{\lambda} = 1\,, \qquad \chi = \tilde{\chi} = 0\,. \tag{10.6}$$

This leaves residual gauge invariances with parameters restricted by

$$\xi = \xi^-(\sigma^-)\,, \qquad \epsilon = \epsilon^-(\sigma_-)$$
$$\tilde{\xi} = \xi^+(\sigma^+)\,, \qquad \tilde{\epsilon} = \epsilon^+(\sigma^+)\,. \tag{10.7}$$

The spinning string action in this gauge, after elimination of P, is

$$I = -T\int dt \oint d\sigma \left\{\partial_+X \cdot \partial_-X + \frac{i}{\sqrt{2}}\left(\psi_- \cdot \partial_+\psi_- - \psi_+ \cdot \partial_-\psi_+\right)\right\}\,. \tag{10.8}$$

The residual gauge invariance is a symmetry of this action, with transformations

$$\delta X = \xi^- \partial_- X + \frac{i}{2}\epsilon^- \psi_-$$

$$\delta \psi_- = \xi^- \partial_- \psi_- + \frac{1}{2}\partial_- \xi^- \psi_- - \epsilon^- \partial_- X \,. \tag{10.9}$$

The Noether charges are precisely $\mathcal{H}_\pm$ and $\mathcal{Q}_\pm$. Upon elimination of P, and using the fact that the $\psi_\pm$ equations of motion implies that

$$\psi'_\pm = \frac{1}{\sqrt{2}}\partial_\pm \psi_\pm \,, \tag{10.10}$$

we find that these Noether charges are

$$\mathcal{Q}_\pm = \pm\frac{T}{2}\partial_\pm X \cdot \psi_\pm \,, \qquad \mathcal{H}_\pm = \frac{T}{2}\left[(\partial_\pm X)^2 \mp \frac{i}{\sqrt{2}}\psi_\pm \cdot \partial_\pm \psi_\pm\right] \,. \tag{10.11}$$

These are the stress tensor and a supercurrent associated with superconformal invariance of the conformal gauge action. They are set to zero by the spinning string constraints.

N.B. The Dirac Lagrangian for a real anticommuting 2D spinor Ψ is

$$L_{Dirac} = -\frac{i}{2}\Psi^T \Gamma^0 \left(\Gamma^+ \partial_+ + \Gamma^- \partial_-\right)\Psi \,, \qquad \Gamma^\pm = \frac{1}{\sqrt{2}}\left(\Gamma^1 \pm \Gamma^0\right) \,. \tag{10.12}$$

For the choice of 2D Dirac matrices $\Gamma^0 = i\sigma_2$ and $\Gamma^1 = \sigma_1$, and 2D spinor components $\Psi^T = (\psi_+, \psi_-)$, we find that

$$L_{Dirac} = \frac{i}{\sqrt{2}}\left(\psi_- \partial_+ \psi_- - \psi_+ \partial_- \psi_+\right) \,. \tag{10.13}$$

This is what we have in the action (10.8) except that the anticommuting worldsheet fields $\psi_\pm$ are also spacetime D-vectors.

10.2 Open spinning string: free ends

The string equations of motion obtained by variation of the spinning string action (10.1) do not extremize this action when the string has endpoints because of a boundary term in the variation. If we assume that the free-end boundary conditions of the NG string still apply, then we find that

$$\delta I|_{\text{on-shell}} = -\frac{i}{2}T \int dt \left[\left(\chi^- \psi_- - \chi^+ \psi_+\right)\cdot \delta X + Te\left(\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+\right)\right]_{\sigma=0}^{\sigma=\pi} \,. \tag{10.14}$$

Since δX is not restricted, we require

$$\left(\chi^- \psi_- - \chi^+ \psi_+\right)_{\text{ends}} = 0 \,, \qquad \left(\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+\right)_{\text{ends}} = 0 \,, \tag{10.15}$$

for both ends. The solution to these requirements is to impose, at each end separately, the boundary conditions

$$\psi_+|_{\text{end}} = \pm\,\psi_-|_{\text{end}} \quad \& \quad \chi^+|_{\text{end}} = \pm\,\chi^-|_{\text{end}} \,. \tag{10.16}$$

The sign at any given end is not significant because we are free to redefine $\psi_\pm \to \pm\psi_\pm$, so we may choose

$$\psi_+|_{\sigma=0} = \psi_-|_{\sigma=0} \,. \tag{10.17}$$

However, the *relative* sign is significant so we have two cases to consider:

- **Ramond sector**. *Same sign boundary conditions*: $\psi_+|_{\sigma=\pi} = \psi_-|_{\sigma=\pi}$. In this case the Fourier series expansions of $\psi_\pm$ are

$$\psi_\pm(t,\sigma) = \frac{1}{\sqrt{2\pi T}} \sum_{k\in\mathbb{Z}} e^{\mp ik\sigma}\, d_k(t) \,. \tag{10.18}$$

  where the d_k are *anticommuting* coefficient functions of t. This obviously satisfies the boundary condition (10.17) at $\sigma = 0$, and the same boundary condition is satisfied at $\sigma = \pi$ because $e^{-ik\pi} = e^{ik\pi}$ for integer k.

- **Neveu-Schwarz sector**. *Opposite sign boundary conditions*: $\psi_+|_{\sigma=\pi} = -\psi_-|_{\sigma=\pi}$. In this case

$$\psi_\pm(t,\sigma) = \frac{1}{\sqrt{2\pi T}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{\mp ir\sigma}\, b_r(t) \,. \tag{10.19}$$

  The NS boundary condition is satisfied because $e^{-ir\pi} = -e^{ir\pi}$ for $r \in \mathbb{Z} + \frac{1}{2}$.

It might now seem that we have two different types of spinning string. However, consistency (modular invariance at one string loop) requires that we include both the Ramond and the Neveu-Schwarz strings as two "sectors" of a single RNS string. We shall now examine these two sectors separately, and obtain the light-cone gauge action for both.

10.2.1 Ramond sector

For the Fourier series expansion (10.18) we have

$$\frac{i}{2}T \int_0^\pi d\sigma \left\{ \psi_- \cdot \dot{\psi}_- + \psi_+ \cdot \dot{\psi}_+ \right\} = \frac{i}{2} \sum_{k\in\mathbb{Z}} d_{-k} \cdot \dot{d}_k = \frac{i}{2} d_0 \cdot \dot{d}_0 + i \sum_{k=1}^\infty d_{-k}\dot{d}_k \,. \tag{10.20}$$

The full Ramond string action in Fourier space is

$$I_R = \int dt \Big\{ \dot{x}^m p_m + \frac{i}{2} d_0 \cdot \dot{d}_0 + i \sum_{k=1}^\infty \left(\frac{1}{k} \alpha_{-k} \cdot \dot{\alpha}_k + d_{-n} \cdot \dot{d}_n \right)$$
$$- \sum_{n\in\mathbb{Z}} \left(\lambda_{-n} L_n + i\chi_{-n} F_n \right) \Big\} \,. \tag{10.21}$$

where

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} [\alpha_{-k} \cdot \alpha_{k+n} + k \, d_{-k} \cdot d_{k+n}] \,, \qquad F_n = \sum_{k \in \mathbb{Z}} \alpha_{-k} \cdot d_{k+n} \,. \tag{10.22}$$

Using the Poisson brackets of the canonical variables, one can show that[39]

$$\{L_m, L_n\}_{PB} = -i \, (m - n) \, L_{m+n} \,,$$
$$\{L_m, F_n\}_{PB} = -i \left(\frac{m}{2} - n \right) F_{m+n} \,,$$
$$\{F_m, F_n\}_{PB} = -2i L_{m+n} \,. \tag{10.23}$$

As for the NG string, the constraints generate gauge invariances via Poisson brackets, and the combination

$$\sum_{n \in \mathbb{Z}} \left(\xi_{-n} L_n + i\epsilon_{-n} F_n \right) \,, \tag{10.24}$$

generates the gauge transformations

$$\delta\alpha_n = -in \left(\xi_n \alpha_0 + i\epsilon_n d_0 \right) - in \sum_{k \neq n} \left(\xi_k \alpha_{n-k} + i\epsilon_k d_{n-k} \right) \,,$$
$$\delta d_n = \epsilon_n \alpha_0 - in\xi_n d_0 + \sum_{k \neq n} \left(\epsilon_k \alpha_{n-k} - in\xi_k d_{n-k} \right) \,. \tag{10.25}$$

We can fix all but the zero-mode gauge invariance with parameter α_0 by the gauge-fixing conditions

$$\alpha_k^+ = 0 \quad (k \neq 0) \,, \qquad d_k^+ = 0 \quad (\forall k) \,. \tag{10.26}$$

A gauge transformation of these conditions yields, assuming that $\alpha_0^+ \neq 0$, the equations $\xi_k = 0$ for $k \neq 0$ and $\epsilon_k = 0$ for all k. We may then solve the equations $L_n = 0$ ($n \neq 0$) and $F_n = 0$ (all n) to get expressions for α_n^- ($n \neq 0$) and d_n^- (all n). These expressions are needed to get the Lorentz generators in terms of transverse canonical variables, but they are not needed for the action, which is

$$S = \int dt \left\{ \dot{x}^m p_m + \frac{i}{2} \mathbf{d}_0 \cdot \dot{\mathbf{d}}_0 + \sum_{k=1}^{\infty} \left(\frac{i}{k} \boldsymbol{\alpha}_{-k} \cdot \dot{\boldsymbol{\alpha}}_k + i\mathbf{d}_{-k} \cdot \dot{\mathbf{d}}_k \right) - \lambda_0 L_0 \right\} \,, \tag{10.27}$$

where, now, recalling that $\alpha_0 = p/\sqrt{\pi T}$ for an open string,

$$L_0 = \frac{p^2}{2\pi T} + \sum_{k=1}^{\infty} \left(\boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k + k \, \mathbf{d}_{-k} \cdot \mathbf{d}_k \right) \,. \tag{10.28}$$

---

[39]Notice that this implies that the F_n are the Fourier components of a worldsheet field of conformal dimension 3/2, at least classically.

From this we see that the mass-shell constraint is $p^2 + \mathcal{M}^2 = 0$ with

$$\mathcal{M}^2 = 2\pi T N \,, \qquad N = N_b + N_f \,, \tag{10.29}$$

where

$$N_b = \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n \,, \qquad N_f = \sum_{k=1}^{\infty} k \, \mathbf{d}_{-k} \cdot \mathbf{d}_k \,. \tag{10.30}$$

In the quantum theory, the transverse oscillator variables have the canonical (anti)commutation relations

$$\left[\alpha_k^I, \alpha_{-k}^J\right] = k\delta^{IJ} \,, \qquad \left\{d_{-k}^I, d_k^J\right\} = \delta^{IJ} \,, \qquad k \geq 0 \tag{10.31}$$

Notice that this includes the anti-commutation relation

$$\left\{d_0^I, d_0^J\right\} = \delta^{IJ} \,, \tag{10.32}$$

which is realised by

$$d_0^I \to \frac{1}{\sqrt{2}} \gamma^I \tag{10.33}$$

where γ^I are matrices spanning the spinor representation of $SO(D-2)$. All physical states of the Ramond string are therefore spinors of $SO(D-2)$,

For $k \neq 0$ we define the oscillator vacuum $|0\rangle$ by

$$\alpha_k|0\rangle = 0 \,, \quad d_k|0\rangle = 0 \,, \qquad k > 0 \,. \tag{10.34}$$

However, the corresponding string states are degenerate; they form a spinor of $SO(D-2)$. Strictly speaking, of the simply-connected group $\text{Spin}(D-2)$, which is a double cover of $SO(D-2)$. Since the spinor of $\text{Spin}(D-2)$ is *not* a representation of $SO(D-1)$, or $\text{Spin}(D-1)$, the quantum theory cannot be Lorentz invariant unless these states are massless. This can also be understood as due to a fermi-bose cancellation of the zero-point energies of the oscillators. This means that, in a basis where the D-momentum operator level number operators are diagonal, that the mass-squared at level $N = N_f + N_b$ is $(2\pi T)N$.

At level-1 we have the states

$$\alpha_{-1}^I|0\rangle \,, \quad d_{-1}^I|0\rangle \tag{10.35}$$

This gives us $2(D-2)$ states, each of which is a spinor of $\text{Spin}(D-2)$, so we have two vector-spinors of $\text{Spin}(D-2)$ and Lorentz invariance requires that they combine to form some tensor-spinor representation of $SO(D-1)$.

There is no immediate restriction on the spacetime dimension D arising from the Ramond string. However, the Neveu-Schwarz sector requires $D = 10$, and since both R and NS sectors are required, we should now restrict to $D = 10$. In that case, the

Ramond ground state is a 16-component spinor of the group Spin(8), which we take to be real because we can choose the matrices γ^I to be real. The 16×16 matrices

$$S_{IJ} = \frac{1}{2}\gamma^{[I}\gamma^{J]} \tag{10.36}$$

obey the commutation relations of the Lie algebra of Spin(8), which is the same as the Lie algebra of $SO(8)$. A special feature of Spin(8) is that its real 16-component spinor representation is reducible. Observe that the matrix

$$\gamma_9 = \gamma^1\gamma^2\cdots\gamma^8 \tag{10.37}$$

has the properties

$$\gamma_9^2 = 1 \qquad \{\gamma_9, \gamma^I\} = 0. \tag{10.38}$$

The latter property implies that γ_9 has zero trace, so we can choose a basis for which

$$\gamma_9 = \begin{pmatrix} \mathbb{I}_8 & 0 \\ 0 & -\mathbb{I}_8 \end{pmatrix}. \tag{10.39}$$

This basis is consistent with reality of the matrices γ^I, so a real 16-component spinor of Spin(8) is the sum of two 8-dimensional representations: the eigenspinors of γ_9 with eigenvalues ± 1; equivalently a chiral and an anti-chiral spinor. In math-speak they are the $\mathbf{8}_s$ (spinor) and $\mathbf{8}_c$ (conjugate spinor) of Spin(8).

To summarise: the ground states of the $D = 10$ Ramond string in light-cone gauge are massless and they transform as the $\mathbf{8}_s \oplus \mathbf{8}_c$ representation of the transverse rotation group. All other states are massive.

10.2.2 Neveu-Schwarz sector

For the Fourier series expansion (10.19) we have

$$\frac{i}{2}T\int_0^\pi d\sigma\left\{\psi_- \cdot \dot{\psi}_- + \psi_+ \cdot \dot{\psi}_+\right\} = \frac{i}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_{-r}\cdot\dot{b}_r = i\sum_{r=\frac{1}{2}}^\infty b_{-r}\dot{b}_r, \tag{10.40}$$

The full NS string action in Fourier space is

$$I_{NS} = \int dt\left\{\dot{x}^m p_m + i\sum_{k=1}^\infty \frac{1}{k}\alpha_{-k}\cdot\dot{\alpha}_k + i\sum_{r=\frac{1}{2}}^\infty b_{-r}\cdot\dot{b}_r\right.$$
$$\left. - \sum_{n\in\mathbb{Z}}\lambda_{-n}L_n - i\sum_{r\in\mathbb{Z}+\frac{1}{2}}\chi_{-r}G_r\right\}, \tag{10.41}$$

where

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_{-k} \cdot \alpha_{k+n} + \frac{1}{2} \sum_{r \in \mathbb{Z}+\frac{1}{2}} r\, b_{-r} \cdot b_{r+n}\,, \qquad n \in \mathbb{Z}$$

$$G_r = \sum_{k \in \mathbb{Z}} \alpha_{-k} \cdot b_{k+r}\,, \qquad\qquad r \in \mathbb{Z}+\frac{1}{2}\,. \tag{10.42}$$

The gauge transformations generated by the linear combination of constraint functions

$$\sum_{n \in \mathbb{Z}} \xi_{-n} L_n + \sum_{r \in \mathbb{Z}+\frac{1}{2}} i\epsilon_{-r} G_r\,, \tag{10.43}$$

are

$$\delta\alpha_n = -in\xi_0\alpha_n - in \sum_{m \neq n} \xi_m \alpha_{n-m} + n \sum_r \epsilon_r b_{n-r}$$

$$\delta b_r = \alpha_0 \epsilon_r + \sum_{s \neq r} \alpha_{r-s}\epsilon_s - ir \sum_m \xi_m b_{r-m}\,. \tag{10.44}$$

Invariance under these gauge transformations may be fixed by an extension of the light-cone gauge condition, as for the Ramond string:

$$\alpha_n^+ = 0 \quad (n \neq 0) \quad \& \quad b_r^+ = 0\,. \tag{10.45}$$

Let's check this: a gauge variation of these conditions yields

$$0 = -in\alpha_0^+ \xi_n\,, \quad (n \neq 0) \qquad \& \quad 0 = \alpha_0^+ \epsilon_r\,. \tag{10.46}$$

This tells us that the gauge invariance is completely fixed except for the ξ_0 transformation (assuming that α_0^+ is non-zero). Having fixed the gauge in this way, we may now solve $L_n = 0$ for α_n^- ($n \neq 0$) and $G_r = 0$ for b_r^-. The resulting expressions for these variables are needed only for the Lorentz generators, not for the gauge-fixed action, which is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \boldsymbol{\alpha}_{-k} \cdot \dot{\boldsymbol{\alpha}}_k + i \sum_{r=\frac{1}{2}}^{\infty} \mathbf{b}_{-r} \cdot \dot{\mathbf{b}}_r - \lambda_0 L_0 \right\}\,, \tag{10.47}$$

where now, recalling that $\alpha_0 = p/\sqrt{\pi T}$ for open strings,

$$L_0 = \frac{p^2}{2\pi T} + \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n + \sum_{r=\frac{1}{2}}^{\infty} r\, \mathbf{b}_{-r} \cdot \mathbf{b}_r\,. \tag{10.48}$$

From this we see that the mass-shell constraint is $p^2 + \mathcal{M}^2 = 0$, with

$$\mathcal{M}^2 = 2\pi T\,(N_b + N_f)\,; \qquad N_b = \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n\,, \quad N_f = \sum_{r=\frac{1}{2}}^{\infty} r\mathbf{b}_{-r} \cdot \mathbf{b}_r\,. \tag{10.49}$$

In the quantum theory, the oscillator variables have the canonical (anti)-commutation relations

$$
\left[\alpha_k^I, \alpha_{-k}^J\right] = k\delta^{IJ}, \qquad k = 1, 2, \ldots
$$
$$
\left\{b_{-r}^I, b_r^J\right\} = \delta^{IJ}, \qquad r = \frac{1}{2}, \frac{3}{2}, \ldots \qquad (10.50)
$$

and the oscillator vacuum $|0\rangle$ is defined by

$$
\alpha_k|0\rangle, \quad k > 0, \qquad b_r|0\rangle = 0, \quad r > 0. \qquad (10.51)
$$

The operators N_b and N_f both annihilate the oscillator vacuum for the ordering as given in (10.49), so in a basis where they are diagonal,

$$
M^2 = 2\pi T \left(N - a\right), \qquad N = N_b + N_f, \qquad (10.52)
$$

where the constant a is now introduced to allow for the ambiguity due to operator ordering. We should no longer expect a bose-fermi cancellation of the zero-point energies because of the different "moding" of the bose and fermi oscillators. In contrast to the Ramond string, the oscillator ground state is non-degenerate, so it corresponds to a scalar particle, which will be a tachyon if $a > 0$.

Let's now look at the first excited states. The operator N_b has integer eigenvalues $0, 1, 2, \ldots$ but the operator N_f has eigenvalues $\frac{1}{2}, \frac{3}{2}, \ldots$, so $2N$ is a non-negative integer, and *the first excited states have $N = \frac{1}{2}$*. These states are

$$
b_{-\frac{1}{2}}^I|0\rangle, \quad I = 1, \ldots, D - 2. \qquad (10.53)
$$

By the same argument that we used at level-1 of the NG string, these must be the polarization states of a massless vector if the quantum theory is to preserve Lorentz invariance. This means that we must now choose

$$
a = \frac{1}{2}. \qquad (10.54)
$$

On the other hand, the zero point energy $-a$ is formally given by

$$
-a = \frac{1}{2}(D - 2)\left[\sum_{n=0}^{\infty}(n + 1) - \sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right)\right]. \qquad (10.55)
$$

The relative sign is due to the opposite sign contribution of bosonic and fermionic oscillators. We can perform the sum using the fact that the generalized ζ-function,

$$
\zeta(s, q) = \sum_{n=0}^{\infty}(n + q)^{-s}, \qquad (10.56)
$$

has a unique value at $s = -1$:

$$
\zeta(-1, q) = -\frac{1}{12}\left(6q^2 - 6q + 1\right). \qquad (10.57)
$$

This gives

$$-a = \frac{1}{2}(D-2)\left[-\frac{1}{12} - \frac{1}{24}\right] = -\frac{(D-2)}{16} \,. \tag{10.58}$$

Using this formula, we see that $a = \frac{1}{2}$ implies that $D = 10$. This value is confirmed by a computation of the Lorentz commutators. The theory is Lorentz invariant only if $D = 10$.

To summarize: Lorentz invariance of the NS string requires $D = 10$ and then we have a scalar tachyon with $M^2 = -\pi T$ at level $N = 0$ and a massless vector at level $N = 1/2$ with physical polarizations in the $\mathbf{8}_v$ (vector) representatiuon of the transverse Spin(8) rotation group. All other states are massive.

11. The superstring

The NS sector of the spinning string still has a tachyon. At the free-string level there would be nothing to stop us from simply discarding the tachyon state; we could have already done that for the NG string. However, we cannot expect arbitrary truncations of the spectrum to be consistent with interactions; anything we throw out will usually reappear in loops in the quantum theory, making the truncation inconsistent. The only way to guarantee consistency of some truncation that removes the tachyon is by means of a symmetry. If there is a symmetry that excludes the tachyon, then its exclusion will be consistent if we can introduce interactions consistent with the symmetry. Nothing of that kind is available for the NG string, or for the NS-sector of the spinning string, but once we combine the NS sector with the R sector, a possibility presents itself: spacetime supersymmetry.

Let's take a closer look at the first three levels of the RNS spinning string:

$$
\begin{array}{lcc}
M^2 = -\pi T : & |0\rangle_{NS} & \mathbf{1} \\[4pt]
M^2 = 0 : & \left\{|0\rangle_R \,, \ b^I_{-\frac{1}{2}}|0\rangle_{NS}\right\} & (\mathbf{8}_s \oplus \mathbf{8}_c) \oplus \mathbf{8}_v \\[4pt]
M^2 = \pi T : & \left\{\alpha^I_{-1}|0\rangle_{NS} \,, \ b^I_{-\frac{1}{2}} b^J_{-\frac{1}{2}}|0\rangle_{NS}\right\} & \mathbf{8}_v \oplus \mathbf{28}
\end{array}
\tag{11.1}
$$

If we assign the anti-commuting NS variables odd "G-parity" and declare that $|0\rangle$ has odd "G-parity' then only states with integer N will survive a physical state condition that requires all NS states to have even G-parity. In particular, this condition will remove the tachyon and the states with $M^2 = \pi T$. We can do the same in the Ramond sector but we should require $|0\rangle$ to have even "G-parity" and we should impose a chirality condition on it. Of the states in the first three levels, only some of the $M^2 = 0$ states survive, and these have the Spin(8) representation content

$$Either : \quad \mathbf{8}_v \oplus \mathbf{8}_c \,, \qquad Or : \quad \mathbf{8}_v \oplus \mathbf{8}_s \,, \tag{11.2}$$

depending on the choice of chirality condition (chiral or anti-chiral). Either choice has the feature that the number of bosons (particles with polarisation in tensor representations) equals the number of fermions (particles with polarisation in spinor representations), as is required by supersymmetry. In fact, we now have the same massless physical states that we would get from the $D = 10$ super-Maxwell theory; remarkably, $D = 10$ is the *maximal* spacetime dimension for which this supersymmetric field theory exists, and it does so only for the minimal number of supersymmetries: $\mathcal{N} = 1$.

It turns out that this prescription for truncating the RNS spinning string (known as the GSO projection) leads to massive supermultiplets of $D = 10$ supersymmetry in all higher levels. With this projection understood, we have the RNS superstring.

11.1 The Green-Schwarz superstring

There is a way to reformulate the RNS superstring that makes manifest its spacetime supersymmetry. It relies on the triality property of the Spin(8) algebra. The representation theory for Spin(8) is unchanged by a permutation of its three 8-dimensional representations. Consider the Ramond string in light-cone gauge: if we make the replacement

$$\left\{ d_k^I; I = 1, \ldots, 8 \right\} \to \left\{ \theta_k^\alpha; \alpha = 1, \ldots, 8 \right\} \tag{11.3}$$

where the θ_k^α are again anticommuting variables, then we have only renamed and relabelled the anticommuting variables, giving them a new spacetime interpretation (as a chiral spinor of the transverse rotation group). Quantizing this Green-Schwarz superstring (in light-cone gauge) less to exactly the same results that we found previously for the Ramond string but with a permutation of the Spin(8) representations

$$\mathbf{8}_v \to \mathbf{8}_s \qquad \mathbf{8}_s \to \mathbf{8}_c, \qquad \mathbf{8}_c \to \mathbf{8}_v. \tag{11.4}$$

The ground state of the GS string has zero mass, like the Ramond string, but the Spin(8) representations at this level are

$$\mathbf{8}_c \oplus \mathbf{8}_v. \tag{11.5}$$

Notice that coincides with the first of the two possibilities (11.2) found from the GSO projection. If we had chosen the new anticommuting variables to be the components of an anti-chiral spinor (i.e. the $\mathbf{8}_c$) then we would have found the second of these two possibilities. In other words, the result of imposing the GSO projection on the RNS string is reproduced by imposing Ramond boundary conditions on the GS superstring, discarding its NS sector. This obviously ensures bose-fermi matching at all levels[40]. To show that one gets supermultiplets at all levels requires more work,

---

[40]the tensor product of $\mathbf{8}_c \oplus \mathbf{8}_v$ with any spin(8) tensor T is $(\mathbf{8}_c \otimes T) \oplus (\mathbf{8}_v \times T)$, which exhibits a bose-fermi matching, and the same is true if T is a tensor-spinor.

which we are not going to do but let's look at the first excited states:

$$\alpha_{-1}|0\rangle_{GS}, \qquad \theta^{\alpha}_{-1}|0\rangle_{GS}. \tag{11.6}$$

This gives us states with $M^2 = 2\pi T$ in the Spin(8) representations

$$(\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) = [(\mathbf{1} \oplus \mathbf{8}_v \oplus \mathbf{28}) \oplus (\mathbf{35}_v \oplus \mathbf{56}_v)] \oplus [(\mathbf{8}_s \oplus \mathbf{56}_c) \oplus (\mathbf{8}_c \oplus \mathbf{56}_s)]$$
$$= [\mathbf{44} \oplus \mathbf{84}] \oplus \mathbf{128}, \tag{11.7}$$

where the representations in the second line are those of Spin(9). Those in the square bracket are the bosons: the $\mathbf{44}$ is a symmetric traceless tensor, which describes a massive spin-2 particle, and the $\mathbf{84}$ is a third-order antisymmetric tensor. All fermions are in the $\mathbf{128}$, which describes a massive spin-3/2 particle; all together we have the massive spin-2 multiplet of $\mathcal{N} = 1$ D=10 supersymmetry.

11.2 Closed superstrings

As for the NG string, the physical states at each level of the closed superstring are just tensor products of two copies of the physical states of the open superstring at that level. In other words, the spin(8) representation content of the *massless* states will be the tensor product of two copies of the spin(8) representations of the massless states of the open string. For the latter we had to choose between the two possibilities of (11.2); the choice didn't matter there but now we have a *relative* choice to make because, for the closed GS superstring we have two sets of oscillators, one for the left-movers and one for the right-movers, and this means that we get two distinct closed superstring theories according to the relative choice of spin(8) chirality for the anticommuting variables $\theta_{\pm}$

- **IIA**. Opposite spin(8) chirality: $(\theta^{\alpha}_{-}, \quad \theta^{\dot{\alpha}}_{+})$. We use $\dot{\alpha} = 1, \ldots, 8$ for the anti-chiral $\mathbf{8}_c$ spinor.

- **IIB**. Same spin(8) chirality: $(\theta^{\alpha}_{-}, \quad \theta^{\alpha}_{+})$.

As Hamlet put it so eloquently: *IIB or not IIB, that is the question.*

In either case we can work out the representation content of the massless states using the following lemma:

- 8×8 **lemma**. Let $i = v, s, c$ label the three 8-dimensional representations of Spin(8). Then

$$\mathbf{8}_i \otimes \mathbf{8}_i = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_i$$
$$\mathbf{8}_i \otimes \mathbf{8}_j = \mathbf{8}_k \oplus \mathbf{56}_k, \qquad (i, j, k \quad \text{cyclic}). \tag{11.8}$$

The $\mathbf{35}_v$ is a 2nd-rank symmetric traceless tensor, the $\mathbf{28}$ is a 2nd-rank antsymmetric tensor, and the $\mathbf{56}_v$ is a 3rd-rank antisymmetric tensor. The $\mathbf{35}_s$ ($\mathbf{35}_c$) is an (anti-)self-dual 4th-rank antisymmetric tensor. The $\mathbf{56}_s$ is a chiral vector-spinor and the $\mathbf{56}_c$ is an anti-chiral vector-spinor.

Using this lemma we find the following results, which we organise according to their RNS origin:

- **IIA string**. The Spin(8) representation content of massless states is

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) = \begin{cases} \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v & \mathrm{NS - NS} \\ \mathbf{8}_v \oplus \mathbf{56}_v & \mathrm{R - R} \\ \mathbf{8}_s \oplus \mathbf{56}_s & \mathrm{R - NS} \\ \mathbf{8}_c \oplus \mathbf{56}_c & \mathrm{NS - R} \end{cases} \tag{11.9}$$

We get a spinor ground state for the Ramond open string, so the fermions of the closed superstring come from its R-NS and NS-R sectors. Notice that these give spinorial spin(8) states of *opposite* chirality. The states in the R-R sector are bi-spinors, which are equivalent to antisymmetric tensors; for the IIA superstring we get a vector A_I and a third-order antisymmetric tensor A_{IJK}.

- **IIB string**.The Spin(8) representation content of massless states is

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) = \begin{cases} \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v & \mathrm{NS - NS} \\ \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_c & \mathrm{R - R} \\ \mathbf{8}_s \oplus \mathbf{56}_s & \mathrm{R - NS} \\ \mathbf{8}_s \oplus \mathbf{56}_s & \mathrm{NS - R} \end{cases} \tag{11.10}$$

The spinorial states from the R-NS sector now have the same chirality as those from the NS-R sector. The R-R states are now a scalar A, a 2nd-order anti-symmetric tensor A_{IJ} and a 4th-order self-dual anti-symmetric tensor A_{IJKL}; the self-duality means that

$$A_{IJKL} = \frac{1}{4!} \epsilon_{IJKLMNPQ} A_{MNPQ} \tag{11.11}$$

where ϵ is the alternating spin(8) invariant tensor.

In both IIA and IIB cases we get the same NS-NS massless states, which are also the same as those of the closed NG string; as we saw in that case, the $\mathbf{35}_v$ can be interpreted as the physical polarisation states of a massless spin-2 particle.

All remaining massless bossing states come from the R-R sector. For example, for the IIA superstring, the full set of tensorial spin(8) representations, coming from the combined NS-NS and R-R sectors, is

$$(\mathbf{1} \oplus \mathbf{8}_v \oplus \mathbf{35}_i) \oplus (\mathbf{288}_v \oplus \mathbf{56}_v) = \mathbf{44} \oplus \mathbf{84} \,, \tag{11.12}$$

where the second equality gives the spin(9) representations. They are the same that we found at the first massive level of the open superstring. In that context the spin(8) representations had to combine into spin(9) representations for consistency

with Lorentz invariance. That's not the case here because we are now dealing with the *massless* particles in the IIA superstring spectrum; in the massless particle context the spin(9) representations are what would be required for Lorentz invariance in *eleven* spacetime dimensions, i.e. D=11. In fact, this is also true for the massless fermions of the IIA superstring.

11.2.1 M-Theory*

Here are a few facts about the fermions: each of the $\mathbf{8}_s \oplus \mathbf{56}_s$ and $\mathbf{8}_c \oplus \mathbf{56}_c$ states are the physical polarisation states of a massless D=10 spin-3/2 particle, either chiral or anti-chiral. Consistency of the interactions of massless spin-3/2 particles *requires* supersymmetry, so their presence in the massless spectrum of the closed spinning string is a simple way of seeing why the GSO projection is necessary for consistency[41]. It follows that the effective D=10 spacetime action for the massless states of either the IIA or the IIB superstring is an $\mathcal{N} = 2$ $D = 10$ supergravity theory. There are two of them, according to whether the two spin-3/2 fields have the same (IIB) or opposite (IIA) chirality.

The maximal spacetime dimension for which a supergravity theory exists is D=11, and dimensional reduction of the unique D=11 supergravity theory to D=10 yields the IIA supergravity theory, which is the effective low-energy theory for the massless states of the IIA superstring. For a long time this was seen as just a coincidence, since superstring theory appeared to require D=10. Another "coincidence" is that the Green-Schwarz construction of a Lorentz covariant and manifestly spacetime supersymmetric action for the IIA and IIB superstrings, also applies to membranes in D=11 (but not to any other p-branes, for any $p > 0$, for any $D > 11$), and its dimensional reduction yields the IIA GS superstring action.

In string theory one can compute, in principle, the amplitude for scattering of any particles in the string spectrum to arbitrary order in a string-loop expansion, with each term being UV finite. However, this expansion is a divergent one; we cannot sum the series, even in principle. This is also typically the case in QFT but the perturbation expansions of QFT are usually derived from an action, and some QFT's can be defined non-perturbatively, e.g. as a continuum limit of a lattice version. String theory is different because all amplitudes are found "on-shell", and a spacetime action constructed order by order from these amplitudes has no more information in it than the computed amplitudes. String theory just gives us a perturbation series; it does not tell us what it is that is being perturbed. The completed non-perturbative theory could be something completely different. The fact that D=10 is the critical dimension of superstring shows only that $D \geq 10$ because some dimensions could be invisible in perturbation theory.

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[41]There is no analogous argument for the open spinning string but quantum consistency of any open string theory requires the inclusion of closed strings.

Fortunately, the constraints due to maximal supersymmetry are so strong that the effective spacetime field theory for the massless particles of the superstring contains a lot of information about non-perturbative string theory, sufficient to show that the five distinct superstring theories[42] are unified by some 11-dimensional theory, known as M-theory, and that this theory includes D=11 supergravity. Unfortunately, we don't really know what this theory is, so it is a bit premature to call it a "theory". The current situation, at the start of the 21st centry, is a little like the situation with quantum theory at the start of the 20th century. As we know, the "old quantum theory" was eventually replaced by Quantum Mechanics. We now need a similar revolution in string/M-theory.

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[42]In addition to the IIA and IIB closed superstring theories we have the Type I open superstring (this is the string theory that results from inclusion of the additional features needed to get quantum consistency of interacting open superstrings) and two heterotic superstring theories (for which the worldsheet action has only $(1,0)$ D=2 supersymmetry).