Symmetries, Fields and Particles. Examples 1.

1. Which of the following processes occur, and by which sort of interactions would they proceed:

$$\bar{\nu}_\mu + p \to \mu^+ + n$$
$$p \to n + e^+ + \nu_e$$
$$\mu^+ \to e^+ + e^- + e^+$$
$$n + K^+ \to \Sigma^+ + \pi^0$$
$$K^+ \to e^+ + \pi^0 + \nu_e$$
$$e^+ + e^- \to \nu_\mu + \bar{\nu}_\mu\,?$$

(For quark content of some of these particles, see Q.2. e^-, μ^- are the electron and muon, each with lepton number 1, and e^+, μ^+ are their antiparticles.)

2. The Coulomb self-energy of a hadron with charge +1 or −1 is about 1 MeV. The quark content and rest energies (in MeV) of some hadrons are

$$n(udd)\ 940\,,\ \ p(uud)\ 938$$
$$\Sigma^-(dds)\ 1197\,,\ \Sigma^0(uds)\ 1192\,,\ \Sigma^+(uus)\ 1189$$
$$\pi^0(u\bar{u} - d\bar{d})\ 135\,,\ K^0(d\bar{s})\ 498\,,\ K^+(u\bar{s})\ 494\,.$$

The u and d quarks make different contributions to the rest energy. Estimate this difference.

3. O(n) consists of $n \times n$ real matrices M satisfying $M^T M = I$. Check that O(n) is a group (product, associativity, identity, inverse). U(n) consists of $n \times n$ complex matrices U satisfying $U^\dagger U = I$. Check similarly that U(n) is a group.

Verify that O(n) and SO(n) are the subgroups of real matrices in, respectively, U(n) and SU(n).

By considering the action of U(n) on $\mathbb{C}^n$, and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, show that U(n) is a subgroup of SO($2n$).

4. Show that for matrices $M \in$ O(n), the first column of M is an arbitrary unit vector, the second is a unit vector orthogonal to the first, ..., the kth column is a unit vector orthogonal to the span of the previous ones, etc. Deduce the dimension of O(n). By similar reasoning, determine the dimension of U(n).

Show that any column of a unitary matrix U is not in the (complex) linear span of the remaining columns.

5. The bracket of (square) matrices X, Y is defined as $[X, Y] = XY - YX$. Show that $[X, Y] = -[Y, X]$, and that for matrices X, Y, Z,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\,.$$

6. Show that any SU(2) matrix U can be expressed in the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Deduce that an alternative form for an SU(2) matrix is

$$U = a_0 I + i\mathbf{a}\cdot\boldsymbol{\sigma}$$

with $(a_0, \mathbf{a})$ real, $\boldsymbol{\sigma}$ the Pauli matrices, and $a_0^2 + \mathbf{a}\cdot\mathbf{a} = 1$. Using the second form, calculate the product of two SU(2) matrices.

7. Let $g(t) = \exp it\sigma_1$. By evaluating $g(t)$ as a matrix, show that $\{g(t) : 0 \le t \le 2\pi\}$ is a 1-parameter subgroup of SU(2). Describe geometrically how this subgroup sits inside the manifold of SU(2).

8. Show that the set of matrices

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 - |\beta|^2 = 1$ forms a group, and that it is a Lie group of dimension 3.

9. Three 3×3 matrices T_i are defined by $(T_i)_{jk} = -\epsilon_{ijk}$. Prove the results

(i) $[T_i\,, T_j] = \epsilon_{ijk} T_k\,,$

(ii) $(\mathbf{a}\cdot\mathbf{T})^3 = -a^2 \mathbf{a}\cdot\mathbf{T}\,,$

(iii) $\exp(\mathbf{a}\cdot\mathbf{T}) = I + \mathbf{a}\cdot\mathbf{T}\,\frac{\sin a}{a} + (\mathbf{a}\cdot\mathbf{T})^2\,\frac{1-\cos a}{a^2}\,,$

where $a = |\mathbf{a}|$.

What are the possible eigenvalues of $\mathbf{n}\cdot\mathbf{T}$ if $\mathbf{n}$ is a unit vector?

10. L(SU(2)) consists of matrices of the form $X = -\frac{1}{2}i\mathbf{x}\cdot\boldsymbol{\sigma}$, with $\mathbf{x}$ real. The adjoint action of $U \in$ SU(2) is $X \to X' = UXU^\dagger$. Show that X' is traceless and antihermitian, so can be written as $X' = -\frac{1}{2}i\mathbf{x}'\cdot\boldsymbol{\sigma}$. Show that $\mathrm{Tr}(X^2) = \mathrm{Tr}(X'^2)$ and express this as a relation between $\mathbf{x}$ and $\mathbf{x}'$.

Deduce that the adjoint action of U can be expressed as $x'_a = R(U)_{ab} x_b$ where $R(U)$ is an orthogonal 3×3 matrix. Why is $R(U)$ in SO(3)? Show that $R(U)_{ab} = \frac{1}{2}\mathrm{Tr}(\sigma_a U \sigma_b U^\dagger)$, and verify that $R(U) = R(-U)$.

11. Verify the Baker–Campbell–Hausdorff formula

$$\exp X \cdot \exp Y = \exp\Big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \dots\Big)$$

to the order shown.

Symmetries, Fields and Particles. Examples 2

1. Consider the element $U = \cos\frac{\alpha}{2}\, I - i\sin\frac{\alpha}{2}\,\sigma_1 \in \mathrm{SU}(2)$. Find U^{-1}. Calculate $U\sigma_a U^{-1}$ for $a = 1, 2, 3$, and deduce that $\mathrm{Ad}\, U$, the rotation in SO(3) corresponding to U, is a rotation by α about the x_1-axis.

2. Let $\exp iH = U$. Show that if H is hermitian then U is unitary. Show also, that if H is traceless then $\det U = 1$. How do these results relate to the theorem that the exponential map $X \to \exp X$ sends L(G), the Lie algebra of G, to G?

3. Show that

$$\exp -\tfrac{1}{2} i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma} = \cos\tfrac{\alpha}{2}\, I - i\sin\tfrac{\alpha}{2}\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}$$

where $\boldsymbol{\alpha} = \alpha\hat{\boldsymbol{\alpha}}$. Deduce that in the region $-\mathrm{Tr}(X^2) < 2\pi^2$, the exponential map $X \to \exp X$ from L(SU(2)) to SU(2) is 1-to-1, and onto almost all of SU(2). Describe the map for $-\mathrm{Tr}(X^2) = 2\pi^2$.

4. For a matrix Lie group G, consider the action of G on itself by conjugation, defined by $g' \to gg'g^{-1}$. Show that the eigenvalues of g' and $gg'g^{-1}$ are the same for all g, so the eigenvalues are invariants of an orbit.

Find the eigenvalues of the SU(2) matrix $\cos\frac{\alpha}{2}\, I - i\sin\frac{\alpha}{2}\,\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}$ where $\boldsymbol{\alpha} = \alpha\hat{\boldsymbol{\alpha}}$. Deduce the orbit structure of SU(2) under the action of SU(2) on itself by conjugation.

5. Consider the two SU(2) elements $g = a_0 I + i\mathbf{a}\cdot\boldsymbol{\sigma}$ with $a_0^2 + \mathbf{a}\cdot\mathbf{a} = 1$, and $g' = b_0 I + i\mathbf{b}\cdot\boldsymbol{\sigma}$ with $b_0^2 + \mathbf{b}\cdot\mathbf{b} = 1$. Recall that $\sigma_i\sigma_j = \delta_{ij} I + i\varepsilon_{ijk}\sigma_k$. Calculate gg' and deduce that the left action of g on SU(2), $g' \to gg'$, defines a 4×4 matrix

$$g_L = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

where the entries $*$ are to be determined. Show that g_L is an O(4) matrix and that the determinant of g_L is $(a_0^2 + \mathbf{a}\cdot\mathbf{a})^2 = 1$, so g_L is an SO(4) matrix.

In this way we have found the subgroup $\mathrm{SU}(2)_L$ of SO(4). By considering elements close to the identity, determine the Lie algebra $\mathrm{L}(\mathrm{SU}(2)_L)$ as a subalgebra of L(SO(4)). Repeat the above calculations for the right action $g' \to g'g^{-1}$, and hence identify the subgroup $\mathrm{SU}(2)_R$ of SO(4), and also the Lie algebra $\mathrm{L}(\mathrm{SU}(2)_R)$ as a subalgebra of L(SO(4)). Show that $\mathrm{L}(\mathrm{SU}(2)_L) \oplus \mathrm{L}(\mathrm{SU}(2)_R) = \mathrm{L}(\mathrm{SO}(4))$, and that elements in the two summands mutually commute. [Hint: Think about the original actions.]

6. Verify that the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}$$

forms a matrix Lie group, G. What is the underlying manifold of G? Is the group abelian? Find the Lie algebra, L(G), and calculate the bracket of two general elements of it.

7. A useful basis for the Lie algebra of GL(n) consists of the n^2 matrices T^{ij} $(1 \leq i, j \leq n)$, where $(T^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Find the structure constants in this basis.

8. Show that if $\mathcal{D}$ is a representation of a Lie group G, acting on a vector space V of dimension N, and A a fixed invertible $N \times N$ matrix, then one may define another (equivalent) representation by the formula

$$\widetilde{\mathcal{D}}(g) = A\, \mathcal{D}(g)\, A^{-1} .$$

Show that $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are related by a change of basis in V. Re-express the formula for $\widetilde{\mathcal{D}}$ in the case that $A = \mathcal{D}(g_0)$ for some fixed element $g_0 \in$ G.

Show that if d is the representation of L(G) associated to $\mathcal{D}$, then $\tilde{d}$, defined by the formula $\tilde{d}(X) = A\, d(X)\, A^{-1}$, is the representation associated to $\widetilde{\mathcal{D}}$. Check that this is a representation. [$\tilde{d}$ is referred to again as a representation of L(G) equivalent to d.]

9. Let $\mathcal{D}$ be a finite-dimensional representation of G acting on V, and $(\ ,\ )$ a positive definite inner product on V invariant under G, i.e.

$$(\mathcal{D}(g)u, \mathcal{D}(g)v) = (u, v) \ : \ u, v \in V\,,\ g \in \mathrm{G}\,.$$

[$\mathcal{D}$ is said to be unitary in this case.]

Let W be an invariant subspace of V. Show that $W_\perp$, the orthogonal complement of W in V, is also invariant.

Deduce that $\mathcal{D}$ is totally reducible to irreducible pieces.

10. (a) Let L be a real Lie algebra (i.e. there is a basis $T_i : i = 1, \ldots, n$ with real structure constants c_{ijk}). Suppose d is a representation of L. Write down the algebraic equations that the matrices $d(T_i)$ must satisfy. Show that the complex conjugate matrices $d(T_i)^*$ also define a representation of L.

(b) Show that the fundamental representation $d(T_a) = -\frac{1}{2}i\sigma_a$ and its complex conjugate $\tilde{d}(T_a) = \frac{1}{2}i(\sigma_a)^*$ are *equivalent* representations of L(SU(2)), with the commutation relations (Lie brackets) $[T_a, T_b] = \varepsilon_{abc}T_c$. Is your matrix A (as in Q.8) in SU(2)? If not, could it be?

Show that the weights of the representations d and $\tilde{d}$ are the same. [The weights of d are the eigenvalues of $i\, d(T_3)$, and similarly for $\tilde{d}$.]

11. (a) Show using the Jacobi identity that the representation ad of L(G), defined by $(\mathrm{ad}\, X)Y = [X, Y]$, is indeed a representation.

(b) Show using the Baker–Campbell–Hausdorff formula that if d is a representation of L(G), then one can sensibly attempt to construct a representation $\mathcal{D}$ of G by the formula $\mathcal{D}(\exp X) = \exp(d(X))$. Could there be problems with this construction?

Symmetries, Fields and Particles. Examples 3

1. Find the weights of the tensor product representation $j \otimes j'$, together with their multiplicities, where j denotes the spin j representation of L(SU(2)), and $j \geq j'$. Deduce the decomposition into irreducibles (Clebsch-Gordan series)

$$j \otimes j' = j + j' \,\oplus\, j + j' - 1 \,\oplus\, \cdots \,\oplus\, j - j' \,.$$

Verify that the dimensions of the two sides are the same.

2.(a) Let $\mathcal{D}$ be a representation of a Lie group G, and d the associated representation of L(G). Show that, for $g \in$ G and $X \in$ L(G),

$$d(gXg^{-1}) = \mathcal{D}(g)d(X)\mathcal{D}(g)^{-1} \,.$$

(b) Let $\mathcal{D}^{(1)} \otimes \mathcal{D}^{(2)}$ be a tensor product representation of G. Show that the associated representation d of L(G) is given by

$$d(X) = d^{(1)}(X) \otimes I + I \otimes d^{(2)}(X) \,.$$

Write out this equation using matrix index notation.

Show that the sums of all pairs of eigenvalues of $d^{(1)}(X)$ and $d^{(2)}(X)$ are eigenvalues of $d(X)$. What can you deduce about the weights of the representation $\mathcal{D}^{(1)} \otimes \mathcal{D}^{(2)}$?

3. Consider a gauge theory with a scalar field $\Phi(x)$ transforming under the representation $\mathcal{D}$ of the gauge group G. Write down how Φ and the gauge potential A_μ transform under a gauge transformation.

Show that the covariant derivative of Φ,

$$D_\mu \Phi = \partial_\mu \Phi + d(A_\mu)\Phi$$

where d is the representation of L(G) associated to $\mathcal{D}$, transforms in the expected way under gauge transformations. Discuss how you may construct a term in the Lagrangian density from $D_\mu \Phi$. What properties of G and its representation $\mathcal{D}$ are important?

4. Write down the formula for the field tensor $F_{\mu\nu}$ of a nonabelian gauge potential A_μ, where the gauge group is G. Also write down how A_μ transforms under a gauge transformation $g(x)$. Determine directly how $F_{\mu\nu}$ gauge transforms.

Show that if $A_\mu = -\partial_\mu g \, g^{-1}$, then $F_{\mu\nu}$ vanishes.
[*Hint*: Use $g\, g^{-1} = I$ to determine $\partial_\mu g^{-1}$.]

5. The Lagrangian density of the abelian Higgs model is

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2}\overline{D_\mu \phi}\, D^\mu \phi - \frac{1}{4}\lambda(\bar{\phi}\phi - v^2)^2 \,,$$

where $D_\mu \phi = \partial_\mu \phi - i a_\mu \phi$. Assume now that the scalar field is expressed as $\phi(x) = e^{i\beta(x)}(v + \eta(x))$, where β and η are real fields. Find the form of $\mathcal{L}$ expressed in terms of these new fields. Determine how these fields transform under a gauge transformation, and verify that $\mathcal{L}$ is still gauge invariant.

Show that β is related to the longitudinal part of a_i.

6. Show that the group U(2) has an SU(2) subgroup, and also a U(1) subgroup whose elements commute with all elements in the SU(2) subgroup. Consider the homomorphism SU(2) $\times$ U(1) $\to$ U(2) defined by $(g, u) \to gu$. Show that this map is onto. Find the kernel (the pairs (g, u) that

map to the identity). Deduce that $\mathrm{U}(2) = (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$. What is the analogous result for $\mathrm{U}(n)$?

Let Φ be a complex 2-component scalar field transforming under the standard action of U(2): $\Phi \to U\Phi$. Let $\Phi_0 = \binom{v}{0}$ with v real and positive. Identify the manifold M that is the orbit of Φ_0 under U(2). Identify the isotropy group of Φ_0, and express M as a homogeneous space. Are you familiar with any other way of regarding this particular manifold M as a homogeneous space?

Can you find other manifolds that can be regarded as homogeneous spaces in more than one way?

7. Schur's Lemma. Let $\mathcal{D}$ be an irreducible representation of a (Lie) group G acting on a vector space V. Let A be an operator acting on V which commutes with the action of G, that is, $A\mathcal{D}(g) = \mathcal{D}(g)A$ for all $g \in \mathrm{G}$. Then $A = \lambda I_V$, where λ is a constant and I_V is the unit operator.

Prove this by showing that any eigenspace of A is an invariant subspace of V, and that there is therefore precisely one eigenspace of A which is the whole of V, and that this gives the desired result.

8. Let $\{T_i\}$ be an adapted basis for a simple Lie algebra L of compact type, so that the structure constants are totally antisymmetric. Let d be an irreducible representation of L, and define the operator Q on L with matrix components

$$Q_{ij} = \mathrm{Tr}(d(T_i)d(T_j))\,.$$

Show that $Q = -\mu I_{\mathrm{L}}$, where $\mu > 0$ and I_{L} is the unit operator on L.
[*Hint*: Use the associativity property of $\mathrm{Tr}([d(T_i), d(T_j)]\, d(T_k))$ to show that $Q\,\mathrm{ad}T_j = (\mathrm{ad}T_j)\,Q$. Verify that the representation ad is irreducible and apply Schur's lemma.]

9. The action for a particle moving on a trajectory $g(t)$ in a compact matrix Lie group G is defined to be

$$S = -\int_{t_0}^{t_1} \mathrm{Tr}(\dot{g}g^{-1}\dot{g}g^{-1})\,.$$

Show that an infinitesimal variation of the trajectory is of the form $\delta g(t) = g(t)\delta X(t)$, where $\delta X(t)$ is in L(G). Using this, show that the Euler-Lagrange equation of motion is

$$\frac{d}{dt}(g^{-1}\dot{g}) = 0\,.$$

Is $\dot{g}g^{-1}$ also time-independent?

Evaluate $g^{-1}\dot{g}$ and $\dot{g}g^{-1}$ for a solution $g(t) = g_0 \exp(tX_0)$.

10. Let $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The group SU(1,1) is defined to be the set of 2×2 matrices U, of determinant 1, satisfying

$$U^\dagger\, \eta\, U = \eta\,.$$

Find the Lie algebra L(SU(1,1)) and select a "standard" basis $\{T_a\}$ for it that is similar to the standard basis of L(SU(2)). Calculate the brackets in your basis, and the structure constants.

Evaluate the Killing form κ_{ab}. Is it nondegenerate? Is it of compact type? Identify a compact subgroup of SU(1,1). Identify a subgroup $\mathbb{R}$ in SU(1,1).

Symmetries, Fields and Particles. Examples 4

1. Consider an arbitrary basis $\{T_i\}$ for a simple Lie algebra L(G). The Killing form $\kappa_{ij} = \kappa(T_i, T_j) = \mathrm{Tr}(\mathrm{ad}\, T_i\, \mathrm{ad}\, T_j)$ is non-degenerate (i.e. invertible). Show that

$$C = -\kappa_{ij}^{-1} T_i T_j$$

is a Casimir operator, satisfying $[C, X] = 0$ for all $X \in \mathrm{L(G)}$.

[*Hint*: Use the associativity property $\kappa(X, [Y, Z]) = \kappa([X, Y], Z)$ to find an identity involving κ_{ij} and the structure constants c_{ijk}, and re-express this in terms of κ_{ij}^{-1} and c_{ijk}.]

2. Matrices in SU(2) can be parametrised as $g = a_0 I + i\mathbf{a}\cdot\sigma$ where a_0 and $\mathbf{a}$ are subject to the constraint $a_0^2 + \mathbf{a}\cdot\mathbf{a} = 1$. Write down g^{-1} and dg, and the constraint satisfied by da_0 and $d\mathbf{a}$. Evaluate $dg\, g^{-1}$ and show that it lies in L(SU(2)).

A Riemannian metric on SU(2), invariant under SU(2) $\times$ SU(2), is

$$-\tfrac{1}{2}\mathrm{Tr}(dg\, g^{-1}\ dg\, g^{-1})\,.$$

Evaluate this in terms of the parameters a_0, $\mathbf{a}$ and their differentials, and show, using the constraints, that it has the form of the standard metric on the unit 3-sphere $ds^2 = da_0^2 + d\mathbf{a}\cdot d\mathbf{a}$.

[*Hint*: Use vector notation throughout. You will need a formula for $(\mathbf{b}\cdot\sigma)(\mathbf{c}\cdot\sigma)$.]

3. Let h_1, h_2, $e_{\pm\alpha}$, $e_{\pm\beta}$, $e_{\pm\gamma}$ be the elements of the Lie algebra of SU(3) defined in lectures.

Use the Jacobi identity to evaluate $[\mathbf{h}, [e_\alpha, e_\beta]]$ where $\mathbf{h} = (h_1, h_2)$, and deduce from the root diagram that $[e_\alpha, e_\beta]$ is proportional to e_γ. Verify by explicit calculation that the constant of proportionality is 1. Study by similar methods $[e_\alpha, e_\gamma]$ and $[e_{-\beta}, e_\beta]$.

4. The weights of the spin j representation of L(SU(2)) are reflection symmetric. By considering this reflection symmetry for the three L(SU(2)) subalgebras of L(SU(3)), show that any finite dimensional representation of L(SU(3)) has similar decompositions into irreducibles under each of the L(SU(2)) subalgebras. Verify this for specific examples.

5. Verify, using L(SU(3)) weights, that $3 \otimes 3 \otimes 3$ decomposes into irreducibles as $10 \oplus 8 \oplus 8 \oplus 1$.

What are the quantum numbers of the SU(3) baryon singlet state, and what is its quark content? How is this state different from the Λ^0 in the baryon octet?

6. An ideal I of a Lie algebra L is defined to be a subalgebra for which $[\mathrm{L}, \mathrm{I}] \subset \mathrm{I}$. A Lie algebra is simple if it is non-abelian and has no proper ideals.

(a) Suppose L is a finite-dimensional Lie algebra of compact type, and I an ideal of L. Let $\mathrm{I}_\perp$ denote the orthogonal complement of I with respect to the Killing form κ. ($Y \in \mathrm{I}_\perp$ if and only if $\kappa(X, Y) = 0$ for all $X \in \mathrm{I}$.) By considering $\kappa(X, [Y, Z])$, where $X \in \mathrm{I}$, $Y \in \mathrm{L}$ and $Z \in \mathrm{I}_\perp$, show that $\mathrm{I}_\perp$ is an ideal and that

$$\mathrm{L} = \mathrm{I} \oplus \mathrm{I}_\perp$$

where the summands mutually commute. Deduce that L can be expressed as a direct sum of simple Lie algebras of compact type.

(b) Show that if L is simple, then

$$[\mathrm{L}, \mathrm{L}] = \mathrm{L}\,. \qquad (*)$$

Verify $(*)$ for L = L(SU(3)), and show that it fails for L(U(3)).

7. Let d be an irreducible representation of L(SU(3)), acting on V. Suppose that v is in the weight space $V_{\underline{\lambda}}$, so that $d(\underline{h})v = \underline{\lambda} v$. Show that $d(e_\alpha)v$, provided it is non-zero, has weight $\underline{\lambda} + \underline{\alpha}$, and

similarly for all the other roots. Deduce that the weights associated with d differ by elements of the root lattice of L(SU(3)), the set $N_1\underline{\alpha} + N_2\underline{\beta}$ with $(N_1, N_2) \in \mathbb{Z}$.

Show that the weight lattice of L(SU(3)) consists of the root lattice together with two independent translates of the root lattice.

Find the three elements in the centre of SU(3), and show using Schur's lemma that in any irreducible representation of SU(3), each element of the centre acts by multiplication by a constant. Characterise the weights in the root lattice and its two translates, by how the centre acts on the corresponding weight spaces.

8. The group SU(3) has a real subgroup SO(3). Identify the Lie algebra of SO(3) as a subalgebra of L(SU(3)), and find a standard basis $\{T_a : a = 1, 2, 3\}$ for this. Show that by a suitable conjugation, iT_3 can be brought to a diagonal form, where it equals $2h_\alpha$. By restriction, any representation d of L(SU(3)) can be regarded as a representation of L(SO(3)). Show that the L(SO(3)) weights are all integral.

Determine how the irreps of L(SU(3)) with dimensions up to 10 decompose into irreps of L(SO(3)). Are your results consistent with what you know about SO(3) tensor products?

9. Suppose G/H is a homogeneous space. G/H is called a symmetric space if there is a Lie algebra decomposition L(G) = L(H) ⊕ M such that

$$
\begin{aligned}
&[\mathrm{L(H)}, \mathrm{L(H)}] \subset \mathrm{L(H)} \\
&[\mathrm{L(H)}, \mathrm{M}] \subset \mathrm{M} \\
&[\mathrm{M}, \mathrm{M}] \subset \mathrm{L(H)}\,.
\end{aligned}
$$

(The second condition says that G/H is a "reductive" homogeneous space and is true rather generally. Why? It is the third condition that is crucial.) Show that with the above conditions, the map M → −M is a symmetry preserving brackets. What is the geometric interpretation of this symmetry on the manifold G/H?

(a) Show that SU(2)/U(1) is a symmetric space.

(b) Show that the Lie algebra of SU(n) is the direct sum of a subspace of real matrices, and a subspace of imaginary matrices. Which of these, if either, is a Lie subalgebra? Using your result, show that SU(n)/SO(n) is a symmetric space.

(c) Show that if all matrices in the subspace of imaginary matrices are multiplied by i, then the direct sum of the two subspaces becomes the matrix Lie algebra L(SL(n;$\mathbb{R}$)). How have the brackets changed?

10. Let L(G) be a simple Lie algebra of compact type with a normalized, adapted basis $\{T_i\}$, so that $\kappa(T_i, T_j) = -\delta_{ij}$, and real structure constants c_{ijk}. Now consider the real Lie algebra Real$\{\mathrm{L(G)}^{\mathbb{C}}\}$ with brackets

$$
[X_i, X_j] = c_{ijk} X_k\,, \quad [X_i, Y_j] = c_{ijk} Y_k\,, \quad [Y_i, Y_j] = -c_{ijk} X_k\,.
$$

Find the Killing form of this algebra, and show that it is not negative definite.

11. Consider the representation of the Lorentz Lie algebra constructed from the Dirac matrices

$$
S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]\,.
$$

Calculate these using the Weyl version of the Dirac matrices. Show that the representation is reducible. Show that in the two irreducible pieces, the rotation generators $J_i = \frac{1}{2}\varepsilon_{ijk}S^{jk}$ are represented in the same way, but that the boost generators $K_i = S^{0i}$ are represented by matrices with opposite signs. Deduce that a Dirac spinor transforms via the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz Lie algebra.

Give an argument showing that the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation is not equivalent to the $(\frac{1}{2}, \frac{1}{2})$ representation.

Part III Symmetries, Fields and Particles (Michaelmas 2014): Example Sheet 1 Solutions

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Revised by Ciaran Hughes and Alex Arvanitakis for 2014.
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General facts

The Pauli sigma matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{0.1}$$

They obey

$$\sigma_i \sigma_j = \delta_{ij} I + i \varepsilon_{ijk} \sigma_k. \tag{0.2}$$

It will also be useful to do some simple manipulations of the totally antisymmetric Levi-Civita tensor, $\varepsilon_{i_1 \dots i_n}$, although in this sheet we only need the case $n=3$. For notational convenience we introduce the same tensor with raised indices but identical numerical value, $\varepsilon^{i_1 \dots i_n}$, then you should convince yourself of the identity

$$\varepsilon_{i_1 \dots i_p i_{p+1} \dots j_n} \varepsilon^{j_1 \dots j_p i_{p+1} \dots i_n} = (n-p)! p! \delta^{j_1 \dots j_p}_{i_1 \dots i_p} \tag{0.3}$$

where the antisymmetrised Kronecker delta is

$$\delta^{j_1 \dots j_p}_{i_1 \dots i_p} = \frac{1}{p!} \left(\delta^{j_1}_{i_1} \dots \delta^{j_n}_{i_p} + \text{signed permutations of } i_1, \dots, i_p \right). \tag{0.4}$$

A given signed permutation has a positive sign if the indices are an even permutation of $1,\dots,p$, and a negative sign if an odd permutation. The expression (0.3) is a relatively intuitive formula, the $(n-p)!$ is there because there are $(n-p)!$ ways of contracting the indices. By definition of the Levi-Civita tensor the remaining indices must all be distinct so that we get a factor of $+1$ or -1, depending on the (antisymmetric) arrangement of indices. The latter feature is of course captured by the antisymmetric combinations of Kronecker deltas (and the $p!$ is just there to cancel the $1/p!$ in the definition of antisymmetrisation.)

In particular,

$$\varepsilon_{ijk} \varepsilon^{ijk} = 3!, \varepsilon_{ijk} \varepsilon^{ljk} = 2! \delta^l_i, \varepsilon_{ijk} \varepsilon^{lmk} = (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i). \tag{0.5}$$

Question 1

As you may have guessed from the course name, and maybe the lectures, this course largely studies symmetries and particles. This initial question tries to get you to think about both of these. For example, say that you were alive during the 20th century and you had an experimentalist friend. This experimentalist friend has done some experiments and found a list of particle decays, some decays which he/she sees often enough to say they actually occur, and others which are not seen and hence have some upper bound for being zero. **Aside**: An experimentalist won't say a decay can absolutely not happen, they will say they haven't seen it in such a long time that it has some upper bound for never decaying, e.g, see the mean life of the proton in the particle data group here:

http://pdg8.lbl.gov/rpp2014v1/pdgLive/Particle.action?node=S016#decayclump_B.

Note that a proton is measured, within current experimental capabilities at least, to never decay within a lifetime of $> 2.1E^{29}$ years according to the particle data group page linked above. Some grand unified theories (which combine the three gauge groups of the standard model into one) predict that the proton actually decays and as such, this is measured to a high level. **End Aside**.

So your experimentalist friend gives you the list of observed decays. As a theorist/phenomenologist you want to describe why nature lets some decays occur but forbids others. As an additional motivation, you want to publish these results and win a nobel prize. See here for nobel prizes in particle physics:

http://www.lhc-closer.es/1/9/2/0

And this is excatly what has happened historically. Some theorist has described symmetries of the theory, which give rise to conserved charges due to Noether's theorem, and these conserved charges, also called conserved quantum numbers, forbid certain decays.

Now you might want to know which symmetries we are concerned with, in this question at least? In every decay:

- Charge, Q, is conserved. The total charge (which is an additive quantum number) of the initial particles has to equal the total charge of the final particles.
- The total initial energy has to equal the total final energy, e.g, energy is conserved in an interaction.
- Baryon number is conserved.
- Total lepton number is conserved.
- Individual family number is conserved.

These are explained more fully on wikipedia. Google is your friend people! Also, how would you know which decays go through which force? Well,

- The coupling of particles to gravity is too weak (by huge orders of magnitude) at the energy scales that we are capable of achieving at current particle colliders (and probably any man-made colliders in the forseeable future), and as such we neglect quantum effects of gravity. We work only on a flat Minkowski background. This is very convenient as gravity is non-renormalisable and as such we can't get predictions out of it using the standard methods such as renormalisation.
- QED is quantum electrodynamics and as such, only involves charged particles. This excludes neutrinos.
- The strong force of nature is described theoretically by Quantum Chromodynamics (QCD), and only involves quarks.
- The weak force involves neutrinos and can change one type of particle to another of a specific type. It also breaks parity.

These forces, except gravity, will be exploreed further in the Standard Model course. Gravity is explored in the general relativity course. More info on the forces of nature can be found here: *http://en.wikipedia.org/wiki/Fundamental_interaction*. Follow the links to the individual forces to get more details. Let's start the question at last!

1. The first process is an allowed weak interaction, the neutrino is a signature that it involves the weak force.
2. The second process is forbidden because the proton is lighter than the neutron (view the decay from the rest frame of the proton). Then energy is not conserved as $m_p = E_i < E_f$.
3. The third process is forbidden by lepton family number conservation. As a side note, due to the fact that neutrinos oscillate in nature (which was only discovered in the 2000's), the neutrinos must have a non-zero mass. This mass however is constrained to be incredible small by previous experiments, see here:
   http://pdg8.lbl.gov/rpp2014v1/pdgLive/Particle.action?node=S066.
   The conserved charge: lepton family number is broken by a term in the Lagrangian which is proportional to the mass of the neutrinos. In nature due to the fact that the neurtinos mass is incredibly small, individual family number is effectively conserved. In theory, it is broken by a miniscule term.
4. The fourth process is an allowed weak interaction. It is a weak interaction as there is a change of quark type, also called a change in quark flavour. However, due to the fact that this decay can also occur through the strong decay $n + K^+ \to p + K^0$, and the fact that strong decays occur faster (see the wikipedia link above about the different forces and their time scales), the above weak decay is never actually seen in nature. Instead the above strong decay is seen.
5. The fifth process is an allowed weak interaction. It has neutrinos.
6. The last process is an allowed weak interaction. Guess how we know it is the weak force?

Question 2

This question is getting you to think about quark content of hadrons, which were only found after the 1960's. We will classify quarks in terms of representation spaces if $SU(3)_{flavour}$ later in the course.

Let Δ_u be the u quark contribution to the rest energy, and Δ_d be the d quark contribution to the rest energy. Then subtracting the Coulomb self-energy from the charged hadrons we can find equations like

$$\Delta_u + 2\Delta_d - 2\Delta_u - \Delta_d = 940 - 937 \tag{2.1}$$

(from subtracting the proton rest energy from the neutron rest energy). This implies $\Delta_d - \Delta_u = 3\text{MeV}$. You can repeat this calculation using the other hadrons listed and then average the results. If you really want.

It should be noted that this question should be taken with a huge grain of salt. There is more happening inside a hadron than just the three quarks and some binding energy. It's not possible to get the quark masses this way. Due to asymtopic freedom (that the QCD coupling is large at small energies) a pion (made from up and/or down quarks/antiquarks) is probably best viewed as a soup of low energy quarks interacting with eachother. Inside this pion, quarks and antiquarks are coming in and out of existence, as well as gluon effects which lead to confinement. Even in heavier hadrons have gluonic effects and quark-antiquark pairs coming in and out of existence which make them difficult to study. If this doesn't convince you, quark mass aren't really physical as you can't measure them directly and they even depend on an energy scale (in the same way a coupling does). This will be covered more in AQFT.

Question 3

Let M, N be orthogonal matrices, $M^TM=I=N^TN$. Then their product MN is also orthogonal, as $(MN)^TMN=N^TM^TMN=N^TN=I$. Note that multiplication is associative. It's obvious that the identity obeys $I^TI=I$. Finally, $M^TM=I\Rightarrow M^TMM^T=M^T$, so $MM^T=I$, and so $M^{-1}=M^T$ and this is also orthogonal. Hence the orthogonal matrices form a group.

The verification that unitary matrices form a group is identical.

If we restrict to real matrices in $\mathrm{U}(n)$ the defining condition $U^\dagger U=I$ becomes $U^TU=I$, so $\mathrm{O}(n)$ must be the subgroup of real matrices in $\mathrm{U}(n)$. We can further check that $U^\dagger U=I\Rightarrow|\det U|^2=1$, so $\det U$ is a unit complex number - the unit complex numbers which are real are ±1, so we do indeed get all of $\mathrm{O}(n)$ this way. Restricting to $\mathrm{SU}(n)$ means taking only matrices whose determinants are 1 and the restriction to real matrices obviously gives $\mathrm{SO}(n)$. As $\det M\det N=\det(MN)$ the product of two matrices in $\mathrm{SO}(n)$ still has unit determinant, so it is indeed a group (and we've already done the work necessary to check the remaining group properties).

Now consider the action of $\mathrm{U}(n)$ on $\mathbb{C}^n\cong\mathbb{R}^{2n}$. Let $v\in\mathbb{C}^n$. The action of $\mathrm{U}(n)$ preserves the norm $v^\dagger v$. If we denote the components of v (in some basis) as $v_i=x_i+iy_i$, for x_i,y_i real, then this norm is

$$|v_1|^2+...|v_n|^2=x_1^2+...x_n^2+y_1^2+...y_n^2. \tag{3.1}$$

The latter is just the usual norm on $\mathbb{R}^{2n}$. The induced action of $\mathrm{U}(n)$ on $\mathbb{R}^{2n}$ leaves this norm invariant and so must constitute a subgroup of the orthogonal group $\mathrm{O}(2n)$ acting on this space. (Alternatively one can directly construct the action of U on $\mathbb{R}^{2n}$, which we do at the end of the solution, below.) To see the restriction to $\mathrm{SO}(2n)$ we argue as follows (this form of argument will be used again).

First, recall that the determinant of an $\mathrm{O}(2n)$ matrix can be either plus one or minus one. Let $R:\mathrm{U}(n)\to\mathrm{O}(2n)$ denote the map expressing an element $U\in\mathrm{U}(n)$ as an element $R(U)\in\mathrm{O}(2n)$. We can take $U=I$, the identity matrix, and it is obvious that $R(I)=I$ and has determinant one. Now, $\mathrm{U}(n)$ is connected meaning that we can reach any element of it from the identity matrix via a continuous path, the map $R(U)$ if expressed explicitly (see below) is continuous, and the determinant too is continuous (as it is polynomial in entries of U). Now, a priori we could have $\det R(U)=\pm1$, however because we know $\det R(I)=+1$ we find that we cannot have $\det R(U)=-1$ without breaking continuity: essentially we are arguing that the determinant cannot jump from $+1$ to -1 at all on $\mathrm{U}(n)$. Thus the above induced action of $\mathrm{U}(n)$ lies always in $\mathrm{SO}(2n)$.

To complete the argument we now show that $\mathrm{U}(n)$ is connected i.e. we can reach any element of it from the identity via a continuous path. Any element $U\in\mathrm{U}(n)$ can always be diagonalised using another unitary matrix P such that

$$U=P\mathrm{diag}(e^{i\theta_1},....e^{i\theta_n})P^{-1} \tag{3.2}$$

(If v is an eigenvector with eigenvalue λ then as U preserves $v^\dagger v$ we have $v^\dagger v=v^\dagger U^\dagger Uv=|\lambda|^2v^\dagger v$ so λ has norm one.) But then

$$U(t)=P\mathrm{diag}(e^{it\theta_1},....e^{it\theta_n})P^{-1}\quad t\in[0,1] \tag{3.3}$$

provides a path in $\mathrm{U}(n)$ connecting U to the identity. So any element of $\mathrm{U}(n)$ can be connected to the identity in this way, and it's a connected group.

Finally, let's exhibit the map $R(U)$ explicitly. Write $U=A+iB$ where A and B are real matrices, and similarly write $v=x+iy$ where x and y are real n-component vectors. Then

$$Uv=(A+iB)(x+iy)=Ax-By+i(Ay+Bx), \tag{3.4}$$

and if we let $z=(x,y)^T\in\mathbb{R}^{2n}$ we can write

$$R(U)z=\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \tag{3.5}$$

which defines $R(U)$. The condition $U^\dagger U=1$ implies that $A^TA+B^TB=1$ and $B^TA-A^TB=0$ and ensures that $R(U)$ is indeed orthogonal.

Question 4

Write the matrix M in terms of n column vectors u_i as $M=(u_1,...,u_n)$. Then the condition $M^TM=I$ is

$$I=\begin{pmatrix}u_1^T\\ \vdots\\ u_n^T\end{pmatrix}\begin{pmatrix}u_1 & ...u_n\end{pmatrix}=\begin{pmatrix}u_1^Tu_1 & u_1^Tu_2 & ... & u_1^Tu_n\\ u_2^Tu_1 & u_2^Tu_2 & & \\ \vdots & & \ddots & \\ & & & u_n^Tu_n\end{pmatrix}. \tag{4.1}$$

Concisely, we need $u_i^Tu_i=1$ for each i, and $u_i^Tu_j=0$ for $i\neq j$, i.e. the first column is some unit vector, the second column is orthogonal to the first and also a unit vector and so on. The orthonormality conditions give a total of n constraints on the columns, while the orthogonality conditions give a total of $\frac{1}{2}n(n+1)-n$ constraints. So altogether there are $\frac{1}{2}n(n+1)$ constraints on the original n^2 entries of the matrix M, meaning that the dimension of O(n) is $n^2-\frac{1}{2}n(n+1)=\frac{1}{2}n(n-1)$.

For U(n) our n columns u_i consist of complex numbers and we have the constraints $u_i^\dagger u_i=1$ for each i. This is a total of n real constraints. We also have $u_i^\dagger u_j=0$ for $i\neq j$, which is a total of $2\left(\frac{1}{2}n(n+1)-n\right)$ real constraints (that is, we have to require both the real and imaginary parts of $u_i^\dagger u_j$ be zero). The original number of real components of the matrix was $2n^2$, and we subtract the total number of constraints from this to find the real dimension of U(n) is $2n^2-n-2\left(\frac{1}{2}n(n+1)-n\right)=n^2$.

Aside: There is another way to do this, and is perhaps easier for more complicated groups. There are two steps involved. Step one: Find the total number of variables of the group. For U(n) this is $2n^2$ real variables. Step two: Find the condition that defines the group, e.g, for U(n) this is $U^\dagger U=1$. Now write $H=U^\dagger U$ and notice H is hermitian. A hermition matrix has n^2 d.o.f. Setting $H=U^\dagger U=1$ in the defining unitary condition gives n^2 constraints. Hence the number of degrees of freedom must be the number of real entries in U, which is $2n^2$, minus the number of constraints, which is n^2, from the above argument. Test the argument out on the orthogonol group above. **End Aside.**

The last part of the question follows as a consequence of the fact in linear algebra that the columns of any invertible matrix are linearly independent, and a unitary matrix is invertible by definition. Alternatively one can show this directly from the conditions $u_i^\dagger u_i=1$ and $u_i^\dagger u_j=0$. Explicitly, suppose for some i we could write $u_i=\sum_{j\neq i}c_ju_j$, where $c_j\in\mathbb{C}$. Then we have $0=u_j^\dagger u_i=\sum_{k\neq i}c_ku_j^\dagger u_k=c_j$ so all the c_i are zero and u_i is zero, which is a contradiction.

Question 5

The first part is obvious. The second part is an uninteresting verification - just write out the commutators and all the terms cancel amongst themselves.

Question 6

In order for U to be special unitary, $U^\dagger U=1$ and $\det U=1$. Hence if

$$U=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\quad \alpha\delta-\beta\gamma=1 \tag{6.1}$$

then the condition $U^{-1}=U^\dagger$ can be written

$$\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}=\begin{pmatrix}\alpha^* & \gamma^*\\ \beta^* & \delta^*\end{pmatrix} \tag{6.2}$$

and solving this for γ,δ shows that a general SU(2) matrix U has the form

$$U=\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}, |\alpha|^2+|\beta|^2=1. \tag{6.3}$$

To deduce the decomposition in terms of Pauli matrices, let $\alpha=a_0+ia_3$ and $\beta=a_2+ia_1$, where the a_μ are real numbers obeying $\sum_\mu a_\mu^2=1$. Then

$$U=\begin{pmatrix}a_0+ia_3 & a_2+ia_1\\ -a_2+ia_1 & a_0-ia_3\end{pmatrix} \tag{6.4}$$

which is clearly in the form

$$U=a_0I+i\vec{a}\cdot\vec{\sigma}. \tag{6.5}$$

Now take this matrix and a second SU(2) matrix $U'=b_0I+i\vec{b}\cdot\vec{\sigma}$. We have

$$UU'=a_0b_0I+ib_0\vec{a}\cdot\vec{\sigma}+ib_0\vec{a}\cdot\vec{\sigma}-\vec{a}\cdot\vec{\sigma}\vec{b}\cdot\vec{\sigma}. \tag{6.6}$$

Write the last term as $a_i b_j \sigma_i \sigma_j$ and use the identity $\sigma_i\sigma_j = \delta_{ij} I + i\varepsilon_{ijk}\sigma_k$ to find

$$UU' = \left(a_0 b_0 - \vec{a}\cdot\vec{b}\right) I + i\left(b_0\vec{a} + a_0\vec{b} - \vec{a}\times\vec{b}\right)\cdot\vec{\sigma}. \tag{6.7}$$

Question 7

Let $g(t) = \exp(it\sigma_1)$. We write out the power series definition of the exponential explicitly and split this into even and odd terms:

$$g(t) = \sum_{n=0}^{\infty} \frac{1}{n!}(it\sigma_1)^n = \sum_{n=0}^{\infty} \frac{1}{2n!}(it\sigma_1)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(it\sigma_1)^{2n+1}. \tag{7.1}$$

Now, $\sigma_1^2 = I$ and $i^{2n} = (-1)^n$, so this means

$$g(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} I + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}\sigma_1 \tag{7.2}$$

hence

$$g(t) = \cos t I + i \sin t \sigma_1 = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}. \tag{7.3}$$

To verify this is a subgroup we first check closure, using $e^A e^B = e^{A+B}$ for A, B commuting,

$$g(t)g(s) = \exp(it\sigma_1)\exp(is\sigma_1) = \exp(i(t+s)\sigma_1) = \begin{cases} g(t+s) & t+s < 2\pi \\ g(t+s-2\pi) & t+s > 2\pi \end{cases}. \tag{7.4}$$

The identity is clearly $g(0)$, and the inverse must be given by $g(t)^\dagger$, which is

$$g(t)^\dagger = e^{-it\sigma_1} = e^{i(2\pi - t)\sigma_1} = g(2\pi - t). \tag{7.5}$$

Associativity follows from associativity of matrix multiplication.

Owing to the identification $g(t) = g(t+2\pi)$ we see that this subgroup, which is isomorphic to U(1) forms a circle S^1 inside the manifold of SU(2), which is isomorphic to the 3-sphere S^3, as can be seen from the previous question where we parametrised a general element of SU(2) in terms of 4 real numbers a_μ such that $\sum_\mu (a_\mu)^2 = 1$. (More entertainingly, this U(1) circle can be identified as the fibre of the Hopf fibration, a famous fibre bundle description of S^3 as an S^1 fibred over an S^2.)

Question 8

We shall show that the group in question is actually SU(1,1), the generalised unitary group associated with a hermitian inner product of signature (1,1). Then the group axioms and dimensionality follow from the argument given in answer 3.

Consider the bilinear form η given as a matrix

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.1}$$

We will determine the form of the matrices

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{8.2}$$

which preserve this, in the sense

$$\eta = U\eta U^\dagger \tag{8.3}$$

which also have determinant 1. Note that in this case $U^{-1} \neq U^\dagger$. U^{-1} for such a matrix takes the simple form

$$\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \tag{8.4}$$

while $U^\dagger$ is

$$\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \tag{8.5}$$

If we rewrite the defining relation as

$$\eta U^\dagger = U^{-1}\eta \tag{8.6}$$

we simply get

$$\delta = \alpha^* \tag{8.7}$$

$$\gamma = \beta^* \tag{8.8}$$

or

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}. \tag{8.9}$$

We have just proved that the group defined by matrices of this form is SU(1,1). That this is a Lie group then follows from the argument in question 3.

Question 9

Let $(T_i)_{jk} = -\varepsilon_{ijk}$. Consider

$$\begin{aligned}[][T_i,T_j]_{kl} &= \varepsilon_{ikm}\varepsilon_{jml} - \varepsilon_{jkm}\varepsilon_{iml} \\ &= -\varepsilon_{ikm}\varepsilon_{jlm} + \varepsilon_{jkm}\varepsilon_{ilm}.\end{aligned} \tag{9.1}$$

We now use the identity $\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ to rewrite this as

$$-\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} + \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} = \varepsilon_{ijm}\varepsilon_{lkm} = \varepsilon_{ijm}(-\varepsilon_{mkl}) \tag{9.2}$$

which means that

$$[T_i,T_j]_{kl} = \varepsilon_{ijm}(T_m)_{kl} \tag{9.3}$$

as asked for. This means that the matrices T_i obey the commutation relations of the Lie algebra of SO(3) or equivalently SU(2).

Now consider $\vec{a}\cdot\vec{T} = a_i T_i$. We have

$$(\vec{a}\cdot\vec{T})^2_{kl} = a_i\varepsilon_{ikm}a_j\varepsilon_{jml} = -a_i a_j(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) \tag{9.4}$$

and

$$(\vec{a}\cdot\vec{T})^3_{ql} = a_p a_i a_j \varepsilon_{pqk}(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) = |\vec{a}|^2 a_p \varepsilon_{pql} - a_p a_l a_k \varepsilon_{pqk}. \tag{9.5}$$

The second term vanishes because we are contracting the symmetric $a_p a_k$ with the antisymmetric ε_{pqk}. Hence we indeed find

$$(\vec{a}\cdot\vec{T})^3_{ij} = -|\vec{a}|^2(\vec{a}\cdot\vec{T})_{ij}. \tag{9.6}$$

Next we consider

$$\exp(\vec{a}\cdot\vec{T}) = 1 + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\vec{a}\cdot\vec{T})^{2n+1} + \sum_{n=1}^{\infty}\frac{1}{2n!}(\vec{a}\cdot\vec{T})^{2n}. \tag{9.7}$$

Here we have again split up the exponential into odd and even sums. The goal now is to iterate the relation between $(\vec{a}\cdot\vec{T})^3$ and $(\vec{a}\cdot\vec{T})$ to simplify these sums. We have (letting $a \equiv |\vec{a}|$)

$$(\vec{a}\cdot\vec{T})^{2n+1} = -a^2(\vec{a}\cdot\vec{T})^{2n-1} = \cdots = (-1)^n a^{2n}(\vec{a}\cdot\vec{T}) \tag{9.8}$$

and

$$(\vec{a}\cdot\vec{T})^{2n} = (-1)^{n-1}a^{2(n-1)}(\vec{a}\cdot\vec{T})^2. \tag{9.9}$$

Hence we have

$$\exp(\vec{a}\cdot\vec{T}) = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}a^{2n+1}\frac{1}{a}(\vec{a}\cdot\vec{T}) - \sum_{n=1}^{\infty}\frac{(-1)^n}{2n!}a^{2n}\frac{1}{a^2}(\vec{a}\cdot\vec{T})^2 = 1 + (\vec{a}\cdot\vec{T})\frac{\sin a}{a} + (\vec{a}\cdot\vec{T})^2\frac{1-\cos a}{a^2}. \tag{9.10}$$

Finally, consider $\vec{n}\cdot\vec{T}$ where $\vec{n}$ is a unit vector, i.e. $|\vec{n}|^2 = 1$. This means that

$$(\vec{n}\cdot T)^3 = -(\vec{n}\cdot\vec{T}). \tag{9.11}$$

Hence eigenvalues λ obey

$$\lambda^3 + \lambda = 0 \Rightarrow \lambda(\lambda+i)(\lambda-i) = 0 \tag{9.12}$$

so the possible eigenvalues are $\lambda = 0, \pm i$.

Question 10

Let $X = -\frac{1}{2}i\vec{x}\cdot\vec{\sigma}$. As the Pauli matrices are traceless and hermitian, $\mathrm{tr}X = 0$ and $X^\dagger = -X$. We have $X' = UXU^\dagger$ for $U \in \mathrm{SU}(2)$. Then

$$\mathrm{tr}X' = \mathrm{tr}(UXU^\dagger) = \mathrm{tr}(U^\dagger UX) = \mathrm{tr}X = 0 \tag{10.1}$$

using the cyclic property of the trace. Also

$$(X')^\dagger = UX^\dagger U^\dagger = -UXU^\dagger = -X' \tag{10.2}$$

so X' is antihermitian. Any traceless antihermitian matrix has the general form

$$\begin{pmatrix} iz_3 & z_1 + iz_2 \\ -z_1 + iz_2 & -iz_3 \end{pmatrix}, \tag{10.3}$$

where the z_i are real, and this obviously can be decomposed in terms of the Pauli matrices, so $X' = -\frac{1}{2}i\vec{x'}\cdot\vec{\sigma}$. Now, we also have

$$\mathrm{tr}(X'^2) = \mathrm{tr}(UXU^\dagger UXU^\dagger) = \mathrm{tr}(X^2) \tag{10.4}$$

using cyclic property again. We note that

$$X^2 = -\frac{1}{4}\vec{x}\cdot\vec{\sigma}\vec{x}\cdot\vec{\sigma} = -\frac{1}{4}|\vec{x}|^2 I \tag{10.5}$$

using $\sigma_i\sigma_j=\delta_{ij}I+i\varepsilon_{ijk}\sigma_k$. So $\mathrm{tr}(X^2)=-\frac{1}{2}|\vec{x}|^2$ and we see that we have

$$|\vec{x}|^2=|\vec{x'}|^2. \tag{10.6}$$

The adjoint action we are considering acts linearly on X, and hence can be expressed directly on $\vec{x}$ as a linear map

$$x'_a=R(U)_{ab}x_b. \tag{10.7}$$

As we have just seen that the norm of $\vec{x}$ is preserved the matrix $R(U)$ must be orthogonal. Now because $U=I$ is an allowed transformation we have that $R(I)=I$ is one of these orthogonal matrices. In the next part of the question we explicitly construct the matrix elements of $R(U)$ as polynomial functions of the elements of U, showing that the map $R(U)$ is a continuous map from the connected group SU(2) to the orthogonal matrices. As the determinant map is also continuous (it is a polynomial in the elements of the matrix $R(U)$), and $\det R(I)=1$ this means that in fact $R(U)$ must always have $\det R(U)=1$. The argument is that $\det R(U)$ could in principle be $+1$ or -1, however as the identity is a possible $R(U)$ with determinant $+1$, it is not possible to continuously vary U and arrive at an $R(U)$ with determinant -1, and this holds for all $U\in\mathrm{SU}(2)$.

Finally, we use the sigma matrix identity $\mathrm{tr}(\sigma_a\sigma_b)=2\delta_{ab}$ to write

$$x_a=i\,\mathrm{tr}(\sigma_a X) \tag{10.8}$$

so

$$x'_a=i\,\mathrm{tr}(\sigma_a X')=i\,\mathrm{tr}(\sigma_a U X U^\dagger)=\frac{1}{2}\mathrm{tr}(\sigma_a U\sigma_b U^\dagger)x_b \tag{10.9}$$

which tells us that

$$R(U)_{ab}=\frac{1}{2}\mathrm{tr}(\sigma_a U\sigma_b U^\dagger) \tag{10.10}$$

from which it is obvious that $R(U)=R(-U)$.

Aside: the latter fact shows that the single element $R(U)$ of SO(3) is associated to two elements $\pm U$ of SU(2). We say that SU(2) is the *double cover* of SO(3). This fact is important for instance in constructing the spinor representations of SO(3). These representations are double-valued as representations of the rotation group (you probably have heard that if you rotate a spinor by 2π it comes back to minus itself) but single-valued as representations of the double cover.

Question 11

We want to show

$$\exp(tX)\exp(tY)=\exp\left(tX+tY+\frac{t^2}{2}[X,Y]+\frac{t^3}{12}[X,[X,Y]]-\frac{t^3}{12}[Y,[X,Y]]\right), \tag{11.1}$$

where we have introduced a real number t as a bookkeeping device to keep track of what order we are working at. The most straightforward way to proceed is a brute force expansion. So on the lhs we have (throwing away anything higher than t^3)

$$\begin{aligned}\exp(tX)\exp(tY)&=\left(1+tX+\frac{1}{2}t^2X^2+\frac{1}{6}t^3X^3\right)\left(1+tY+\frac{1}{2}t^2Y^2+\frac{1}{6}t^3Y^3\right)\\&=1+t(X+Y)+\frac{1}{2}t^2(X^2+Y^2+2XY)+\frac{1}{2}t^3(XY^2+X^2Y)+\frac{1}{6}t^3(X^3+Y^3).\end{aligned} \tag{11.2}$$

On the rhs we have

$$\begin{aligned}&1+t(X+Y)+\frac{1}{2}t^2[X,Y]+\frac{1}{12}t^3(X[X,Y]-[X,Y]X+Y[Y,X]-[Y,X]Y)+\frac{1}{2}\left(t(X+Y)+\frac{1}{2}t^2[X,Y]\right)^2+\frac{1}{6}t^3(X+Y)^3\\&=1+t(X+Y)+\frac{1}{2}t^2\left(XY-YX+X^2+XY+YX+Y^2\right)\\&+t^3\left(\frac{1}{12}[X[X,Y]-[X,Y]X+Y[Y,X]-[Y,X]Y]+\frac{3}{12}(X[X,Y]+Y[X,Y]+[X,Y]X+[X,Y]Y)\right.\\&\left.+\frac{2}{12}\left(X^3+Y^3+XYX+X^2Y+XY^2+YX^2+Y^2X+YXY\right)\right)\end{aligned} \tag{11.3}$$

We just need to tidy up the t^3 terms, namely

$$\begin{aligned}&X[X,Y](1/12+3/12)+[X,Y]X(-1/12+3/12)+Y[Y,X](1/12-3/12)+[Y,X]Y(-1/12-3/12)\\&+\frac{2}{12}\left(XYX+X^2Y+XY^2+YX^2+Y^2X+YXY\right)\\&=\frac{1}{12}\left(4X^2Y-4XYX+2XYX-2YX^2-2Y^2X+2YXY-4YXY+4XY^2\right.\\&\left.+2XYX+2X^2Y+2XY^2+2YX^2+2Y^2X+2YXY\right)\\&=\frac{1}{2}(X^2Y+XY^2).\end{aligned} \tag{11.4}$$

Comparing, we find that we are done.

Part III Symmetries, Fields and Particles (Michaelmas 2013): Example Sheet 2 Solutions

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Updated: November 19, 2014.

Please send me comments and corrections.

Question 1

From question 7 of the first example sheet we know we can write $U = e^{-i\alpha\sigma_1/2}$, so $U^{-1} = e^{i\alpha\sigma_1/2}$. We obviously have $\sigma_1\sigma_1\sigma_1 = \sigma_1$, and

$$\sigma_1\sigma_2\sigma_1 = i\sigma_3\sigma_1 = -\sigma_2 \ , \ \sigma_1\sigma_3\sigma_1 = i\sigma_1\sigma_2 = -\sigma_3 \, . \tag{1.1}$$

Then $U\sigma_1U^{-1} = \sigma_1$, while for σ_2 we have

$$\left(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\sigma_1\right)\sigma_2\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\sigma_1\right) = \sigma_2\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) + i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}(\sigma_2\sigma_1 - \sigma_1\sigma_2) \tag{1.2}$$

so

$$U\sigma_2U^{-1} = \cos\alpha\sigma_2 + \sin\alpha\sigma_3 \, . \tag{1.3}$$

Entirely similarly we get

$$U\sigma_3U^{-1} = \cos\alpha\sigma_3 - \sin\alpha\sigma_2 \, . \tag{1.4}$$

Therefore for $x_a = (x_1, x_2, x_3)$, $x_a\sigma_a$ is rotated into $x'_a\sigma_a$ where

$$x'_a = (x_1, \cos\alpha x_2 - \sin\alpha x_3, \cos\alpha x_3 + \sin\alpha x_2) \tag{1.5}$$

so we have the following rotation in SO(3):

$$\mathrm{Ad}(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} . \tag{1.6}$$

This is of course a rotation by α about the x_1 axis.

Question 2

Let $e^{iH} = U$. If H is hermitian, then $H^\dagger = H$, so that $U^\dagger U = e^{-iH^\dagger}e^{iH} = e^{-iH}e^{iH} = e^{i(H-H)} = I$, so U is unitary. Next consider $\det U = \det e^{iH}$. Suppose H has eigenvalues λ_j, then e^{iH} has eigenvalues $e^{i\lambda_j}$. As the determinant is the product of the eigenvalues and the trace is the sum, we have $\det e^{iH} = \prod_j e^{i\lambda_j} = e^{i\sum_j \lambda_j} = e^{i\mathrm{tr}H}$. Thus if H is traceless then $\det U = 1$.

These results are useful when considering the Lie algebras of for instance SO(n), SU(n) and so on. For instance the Lie algebra of SU(n) can be taken to consist of traceless antihermitian matrices iH, and we see that the reason this is so is dictated by the above results.

Question 3

The first part of this questions follows from the fact that $(\vec{\alpha}\cdot\vec{\sigma})^2 = \alpha^2 I$, and is essentially the same computation as the exponential in question 7 of the first example sheet.

For the second part, note that for $X = -\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}$ we have $-\mathrm{tr}(X^2) = \frac{1}{2}\alpha^2$. The region $-\mathrm{tr}(X^2) < 2\pi^2$ corresponds to $\alpha^2 < 4\pi^2$ or $\alpha < 2\pi$ (α is the norm of a vector and so positive). Now any element of SU(2) has the form $a_0 I + i\vec{a}\cdot\vec{\sigma}$ with $\sum_\mu (a_\mu)^2 = 1$. Now,

$$e^X = \cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}\hat{\alpha}\cdot\vec{\sigma} \tag{3.1}$$

and so this defines a map into SU(2) where for the real coordinates a_μ we have

$$a_0 = \cos\frac{\alpha}{2} \quad \vec{a} = -\sin\frac{\alpha}{2}\frac{\vec{\alpha}}{\alpha} \tag{3.2}$$

Suppose that $\vec{\alpha}$ and $\vec{\beta}$ map to the same element of SU(2), for $\alpha, \beta < 2\pi$. Then by matching coefficients we have $\cos\frac{\alpha}{2} = \cos\frac{\beta}{2}$, which implies $\alpha = \beta$, and from this it follows that indeed $\vec{\alpha} = \vec{\beta}$, so it is one-to-one in for $\alpha < 2\pi$. It is also (I think) clear that this is onto "almost all" of SU(2) as given a_μ such that $(a_0)^2 + \sum_i (a_i)^2 = 1$ we know that each a_μ has norm less than one, and so can be written in the form of e^X given above. We also know that SU(2) is connected and therefore the exponential map must be surjective. The reason this is only "almost all" comes about by considering the last part of the question. The condition $-\text{tr}(X^2) = 2\pi^2$ corresponds to $\alpha = 2\pi$, and for this value of α we have $e^X = \cos\pi I = -I$. We see that an entire 2-sphere of radius $\alpha = 2\pi$ is mapped to the single element $-I$ of SU(2), and thus the exponential map is no longer one-to-one. (This can be contrasted with the other case where we don't have a piece involving the sigma matrices - the single point $\alpha = 0$ maps uniquely to the element I of SU(2).)

Question 4

This is a basic fact from linear algebra. Consider for instance that the eigenvalues are defined by the characteristic polynomial $\det(g' - \lambda I) = 0$. We also have $\det(gg'g^{-1} - \lambda I) = \det(gg'g^{-1} - \lambda g I g^{-1}) = \det g(g' - \lambda I)g^{-1}) = \det(g' - \lambda I)$, so we have the same characteristic equation and hence same eigenvalues λ.

Next consider the SU(2) matrix $e^{-\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}}$. As $\left(-\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right)^2 = -\frac{1}{4}\alpha^2$, $(-\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma})$ has eigenvalues $\pm\frac{i}{2}\alpha$. As a result the eigenvalues of the SU(2) matrix are $e^{\pm\frac{i}{2}\alpha}$. (Note that both eigenvalues occur with multiplicity one (for $\alpha \neq 0, 2\pi$) so that the product of the eigenvalues is one, as required by the determinant condition.) As eigenvalues are preserved under conjugation, the orbits of SU(2) are the conjugacy classes $C_\alpha = \{X \in \text{SU}(2) | X \text{ has eigenvalues } e^{\pm i\alpha/2}, \alpha \in [0, 2\pi]\}$.

Question 5

Let $g = a_0 I + i a_i \sigma_i$, $g' = b_0 I + i b_i \sigma_i$, then (as we did in the previous example sheet)

$$gg' = (a_0 b_0 - a_i b_i)I + i\left(a_0 b_i + b_0 a_i - \varepsilon_{ijk} a_j b_k\right)\sigma_i\,. \tag{5.1}$$

We view this as a linear map acting on the four component vector (b_0, b_1, b_2, b_3). Then by evaluating each component of gg' it is easy to see this linear map is given by

$$g_L = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}. \tag{5.2}$$

Using the condition $(a_0)^2 + a_i a_i = 1$ it is easy to check that this satisfies $g_L^T g_L = I$, so is an element of O(4). Calculating a four-by-four determinant doesn't sound fun, so you could just use the following Mathematica code:

```
A = {{ a0 , -a1, -a2, -a3}, {a1, a0, a3, -a2}, {a2, -a3, a0, a1}, {a3,
    a2, -a1, a0}}
Simplify[Det[A]]
```

which shows that $\det g_L = (a_0^2 + \vec{a}\cdot\vec{a})^2$ indeed. Hence it's really an element of SO(4). Alternatively one could note that as $gg' \in \text{SU}(2)$ and SU(2) is isomorphic to the three-sphere in four-dimensional space, then the map $g' \to gg'$ is an isometry of the three-sphere and so an element at least of O(4), and then of SO(4) by arguments about connectivity and the determinant, like we encountered in the previous sheet.

Considering elements close to the identity means we now suppose a_1, a_2, a_3 are infinitesimal, so that then $a_0 = \sqrt{1 - \vec{a} \cdot \vec{a}} \approx 1 + O(a_i)^2$, and we have a group element

$$\begin{pmatrix} 1 & -a_1 & -a_2 & -a_3 \\ a_1 & 1 & a_3 & -a_2 \\ a_2 & -a_3 & 1 & a_1 \\ a_3 & a_2 & -a_1 & 1 \end{pmatrix} \tag{5.3}$$

The bit of this not involving the identity matrix gives us a general Lie algebra element times the infinitesimal parameters. So we find the Lie algebra of $L(\mathrm{SU}(2)_L)$ is given by the span of the following matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.4}$$

For calculational purposes it is convenient to write these in terms of two-by-two blocks as

$$A_1 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{5.5}$$

It is then straightforward to calculate that $[A_i, A_j] = -2\varepsilon_{ijk} A_k$, defining $T_i = -\frac{1}{2} A_i$ we have $[T_i, T_j] = \varepsilon_{ijk} T_k$ which is the standard commutation relations for $L(\mathrm{SU}(2))$.

We can do the same thing for the right action, taking $h = c_0 I + i c_i \sigma_i$ we have

$$g' h^{-1} = (c_0 b_0 + c_i b_i) I + i\,(c_0 b_i - b_0 c_i - \varepsilon_{ijk} c_j b_k) \tag{5.6}$$

giving rise to

$$g_R = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ -c_1 & c_0 & c_3 & -c_2 \\ -c_2 & -c_3 & c_0 & c_1 \\ -c_3 & c_2 & -c_1 & c_0 \end{pmatrix}. \tag{5.7}$$

The calculation goes through in similar fashion. The Lie algebra generators are now

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.8}$$

or

$$C_1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}. \tag{5.9}$$

These again obey $[C_i, C_j] = -2\varepsilon_{ijk} C_k$ so give us another $L(\mathrm{SU}(2))$, as can be checked using properties of the sigma matrices. One can check explicitly that $[A_i, C_j] = 0$ for all i, j. (This result is a consequence of the fact that the left- and right- group actions obviously commute themselves, and this property is inherited by the left- and right- Lie algebras.)

Finally, to exhibit explicitly that $L(\mathrm{SU}(2)_L) \oplus L(\mathrm{SU}(2)_R) = L(\mathrm{SO}(4))$, consider the combinations

$$\frac{1}{2}(A_i \pm C_i) \tag{5.10}$$

which give the "standard" basis of antisymmetric four-by-four matrices, eg

$$\frac{1}{2}(A_1 + C_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ \frac{1}{2}(A_1 - C_1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.11}$$

The Lie algebra of SO(4) of course consists of such matrices. The splitting into independent A_i and C_i parts then constitutes a particular choice of basis for $L(\mathrm{SO}(4))$ which naturally brings out the $L(\mathrm{SU}(2)_L) \oplus L(\mathrm{SU}(2)_R)$ structure. (Essentially what we are doing here is showing that the A_i, C_j give six linearly independent basis vectors for the space of 4 by 4 real antisymmetric matrices.)

Question 6

Consider the matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \ a, b, c \in \mathbb{R}^3 . \tag{6.1}$$

We can check that the product of two such matrices belongs to the same set:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a + a' & b + b' + ac' \\ 0 & 1 & c + c' \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

The group multiplication map $((a, b, c), (a', b', c')) \to (a + a', b + b' + ac', c + c')$ is obviously smooth, as is the map taking an element to its inverse: $a' = -a, b' = -b + ac$, $c' = -c$. Hence this is a matrix Lie group G. The underlying manifold is $\mathbb{R}^3$. The group is not abelian, if we swap the order of the above two matrices it is easy to see we do not get the same product, because of the ac' term in the upper right entry. Lie algebra elements are of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.3}$$

The commutator of two general elements is

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a' & b' \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a' & b' \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac' - a'c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.4}$$

Question 7

We consider the matrices T^{ij} with $T^{ij}_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. The commutator is

$$\begin{aligned} T^{ij}_{\alpha\beta} T^{kl}_{\beta\gamma} - (i \leftrightarrow k, j \leftrightarrow l) &= \delta_{i\alpha}\delta_{j\beta}\delta_{k\beta}\delta_{l\gamma} - (i \leftrightarrow k, j \leftrightarrow l) \\ &= \delta_{jk}\delta_{i\alpha}\delta_{l\gamma} - (i \leftrightarrow k, j \leftrightarrow l) \\ &= \delta_{jk}\delta_{im}\delta_{ln} T^{mn}_{\alpha\gamma} - (i \leftrightarrow k, j \leftrightarrow l) \end{aligned} \tag{7.1}$$

from which we find the structure constants

$$f^{ij,kl}{}_{mn} = \delta_{im}\delta_{jk}\delta_{ln} - \delta_{km}\delta_{il}\delta_{jn} . \tag{7.2}$$

Question 8

Suppose $\mathcal{D}$ is a representation of a Lie group G acting on V, i.e. $\mathcal{D}(g_1g_2) = \mathcal{D}(g_1)\mathcal{D}(g_2)$ for $g_1, g_2 \in G$. Let $\widetilde{\mathcal{D}}(g) = A\mathcal{D}A^{-1}$. Then

$$\widetilde{\mathcal{D}}(g_1g_2) = A\mathcal{D}(g_1g_2)A^{-1} = A\mathcal{D}(g_1)\mathcal{D}(g_2)A^{-1} = \mathcal{D}(g_1)AA^{-1}\mathcal{D}(g_2)A^{-1} = \widetilde{\mathcal{D}}(g_1)\widetilde{\mathcal{D}}(g_2) \tag{8.1}$$

so this also defines a representation. We can obviously view A as defining a change of basis in V, i.e. $v \to \tilde{v} = Av$ for $v \in V$, so that $\mathcal{D}(g)v \to A\mathcal{D}(g)v = A\mathcal{D}(g)A^{-1}Av \equiv \widetilde{\mathcal{D}}\tilde{v}$. If $A = D(g_0)$ then we have $\widetilde{\mathcal{D}}(g) = D(g_0 g g_0^{-1})$.

Now we associate d to $\mathcal{D}$ by expanding $\mathcal{D}(g)$ near the identity, $\mathcal{D}(g) = I + td(X) + O(t^2)$. So we have

$$I + t\tilde{d}(X) + O(t^2) = \widetilde{\mathcal{D}}(g) = A\left(I + td(X) + O(t^2)\right)A^{-1} \tag{8.2}$$

which immediately implies that $\tilde{d}(X) = Ad(X)A^{-1}$. To check this is a representation we compute

$$\tilde{d}([X,Y]) = Ad([X,Y])A^{-1} = A\left(d(X)d(Y) - d(Y)d(X)\right)A^{-1} = A\left(d(X)A^{-1}Ad(Y) - d(Y)A^{-1}Ad(X)\right)A^{-1} = \left[\tilde{d}(X), \tilde{d}(Y)\right], \tag{8.3}$$

as needed.

Question 9

Let $x \in W_\perp$, so that $(u, x) = 0$ for all $u \in W$. Then for $g \in G$ and $u \in W$ we have $(u, \mathcal{D}(g)x) = (\mathcal{D}(g)\mathcal{D}(g)^{-1}u, \mathcal{D}(g)x) = (\mathcal{D}(g)^{-1}u, x) = 0$. In the second to last equality we used the fact that the inner product is invariant under G, and in the last equality we used that $\mathcal{D}(g)^{-1}u \in W$ as W is invariant under G. As a result we see that $x \in W_\perp$ implies $(u, \mathcal{D}(g)x) = 0$ for all $g \in G$ and $u \in U$, hence $\mathcal{D}(g)x \in W_\perp$ and $W_\perp$ is also invariant.

The representation would be reducible if we could find an invariant subspace W. It is totally reducible if it can be decomposed in a block diagonal form. This is clearly possible if the orthogonal subspace $W_\perp$ is invariant (consider the form of a block decomposition of $\mathcal{D}(g)$), and so we have shown that $\mathcal{D}$ is totally reducible to irreducible pieces (for completeness one could consider repeating the process by looking inside each such invariant orthogonal subspace for further reducible subspaces).

Question 10

a) We must have

$$[d(T_i), d(T_j)] = c_{ijk}d(T_k) \tag{10.1}$$

and also the Jacobi identity (which is trivially obeyed by a matrix commutator, of course). We're assuming that the c_{ijk} are real, as a result when we take the complex conjugate of the above equation we get

$$[d(T_i)^*, d(T_j)^*] = c_{ijk}d(T_k)^* \tag{10.2}$$

so the complex conjugate matrices also define a representation.

b) Consider the Lie algebra of SU(2), with $d(T_a) = -\frac{i}{2}\sigma_a$, and the complex conjugate $\tilde{d}(T_a) = \frac{i}{2}(\sigma_a)^*$. Note that $\sigma_1^* = \sigma_1, \sigma_2^* = -\sigma_2$ and $\sigma_3^* = \sigma_3$. It's easy to check that $\sigma_2\sigma_1\sigma_2 = i\sigma_2\sigma_3 = -\sigma_1$ and similarly $\sigma_2\sigma_3\sigma_2 = -\sigma_3$, so combined with $\sigma_2\sigma_2\sigma_2 = \sigma_2$ we have

$$\sigma_2\sigma_a\sigma_2^{-1} = -\sigma_a^*. \tag{10.3}$$

Hence

$$\sigma_2 d(T_a)\sigma_2^{-1} = -\sigma_2\frac{i}{2}\sigma_a\sigma_2 = \frac{i}{2}\sigma_a^* = \tilde{d}(T_a) \tag{10.4}$$

so we have explicitly shown they are equivalent in the sense of question 8. (The commutation relations are easily checked from the sigma matrix identity $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$.) Our matrix A is σ_2 itself:

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{10.5}$$

This matrix is unitary. The determinant however is $-(-i)i = -1$, so it is not in SU(2). However we can take it to be in SU(2) by multiplying by an appropriate phase factor (this is true for any matrix in u(2), and also true for u(n) and su(n)). If we take instead

$$A = e^{i\pi/2}\sigma_2 \tag{10.6}$$

then $\det A = e^{i\pi} \det \sigma_2 = +1$.

We have

$$id(T_3) = \frac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \tag{10.7}$$

$$i\tilde{d}(T_3) = -\frac{1}{2}\sigma_3^* = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{10.8}$$

which clearly have the same eigenvalues, $\pm 1/2$.

Question 11

a) We define ad by $\mathrm{ad}(X)Y = [X, Y]$. To check it gives a representation of the Lie algebra, we compute

$$\mathrm{ad}([X, Y])Z = [[X, Y], Z] \tag{11.1}$$

and

$$[\mathrm{ad}X, \mathrm{ad}Y]Z = \mathrm{ad}(X)(\mathrm{ad}(Y)Z) - \mathrm{ad}(Y)(\mathrm{ad}(X)Z) = [X, [Y, Z]] - [Y, [X, Z]] = -[[Y, Z], X] - [[Z, X], Y] = [[X, Y], Z] \tag{11.2}$$

using in last step the Jacobi identity. So this is indeed a representation.

b) Recall the BCH formula was

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\text{further commutators}}. \tag{11.3}$$

Suppose d is a representation of $L(G)$. We try to define $\mathcal{D}(\exp X) = \exp(d(X))$. To check this is a representation consider

$$\mathcal{D}(\exp X)\mathcal{D}(\exp X) = e^{d(X)}e^{d(Y)} = e^{d(X)+d(Y)+\frac{1}{2}[d(X),d(Y)])+\text{commutators}}. \tag{11.4}$$

Because d is a representation of the Lie algebra we can "pull it out of the commutators", i.e.

$$e^{d(X)+d(Y)+\frac{1}{2}[d(X),d(Y)])+\text{commutators}} = e^{d(X+Y+\frac{1}{2}[X,Y]+\text{commutators})} = \mathcal{D}\left(e^{X+Y+\frac{1}{2}[X,Y]+\text{commutators}}\right) = \mathcal{D}(e^X e^Y), \tag{11.5}$$

using at the last step the BCH formula "in reverse." Thus we have $\mathcal{D}(\exp X)\mathcal{D}(\exp X) = \mathcal{D}(e^X e^Y)$ so this is indeed a representation.

The main problem with this constructions is that we might not necessarily get a representation of the entire group G doing this, as in general the exponential maps $L(G)$ only to the connected component of the identity in G.

Part III Symmetries, Fields and Particles (Michaelmas 2013): Example Sheet 3 Solutions

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Updated: December 1, 2014.

Please send me comments and corrections.

This is a short section with some dialogue which aims to compliment the course material, trying to give some more motivation as to why you would want to learn about tensor products of representations, and also why you would want to study Cartan Algebras.

You might be asking yourself, why go through all this tedious work of finding the tensor product of representions of groups? How is it useful? Well, spherical harmonics $Y_{l,m}(\psi,\phi)$ for $m=-l,\dots,+l$ form the standard basis (for the representation space) of a $(2l+1)$-dimensional irrep in function space of SO(3)[1]. Similarly, we have seen that representation spaces of $SU(2)$ are classified by quantum numbers $|s,m\rangle$. In a quantum mechanical system, for example the hydrogen atom, the orbital part of the wavefunction can be written in terms of the spherical harmoics $Y_{l,m}(\psi,\phi)$, i.e, a basis for the $(2l+1)$-dimensional representation space of $SO(3)$. If the system also has spin, like the hydrogen atom, then the state is also in a $(2s+1)$-dimensional representation space of $SU(2)$.

However, these orbital and spin angular momentum components talk to eachother physcially, and should be combined into a new $(2J+1)$-dimensional irrep of some new group. What is this new group? Well, maybe you can guess with all the work you have done, the new group is a tensor product group of $SO(3)$ and $SU(2)$. The $(2J+1)$-d irrep of the tensor product group, specificcally the tensor product of the $(2l+1)$-d irrep of $SO(3)$ and the $(2s+1)$-d irrep of $SU(2)$, is the total angular momentum irrep where $J\in|l+s|,\dots,|l-s|$. And hence you have found your new symmetry group, and the new conserved quantum number, the total angular momentum J. You can see immediately why the orbital or spin angular momentum are not conserved individually anymore, it is only the tensor product that is a symmetry of the resulting theory.

Now we go onto a seperate discussion. The focus of this discussion is: why you should be interested in the Cartan algebra and the weights of a representation. The answer is because they allow you to describe the states of your theory purely as quantum numbers given by your chosen symmetry. Once you specify a symmetry of your theory, this symmetry allows you to map each basis element in the representation space of your theory to a weight vector in Cartan space. The symmetry you impose will provide strict constraints on which weight vectors are allowed for a particular representaion. An equivalent statement to this is that the weight diagram in Cartan space will be different depending on the symmetry you are considering. Different symmetries place different contraints and hence yield different weights, therefore giving different weight diagrams.

These weight diagrams can be physically useful. For example, since we are forcing a group to be a symmetry of our theory, and each symmetry should leave the Lagrangian, and hence the Hamiltonian, invariant, the energies should be the same for each basis element in the representation space. For $SU(2)$, this allows us to classify spin angular momentum, albeit with a rather trivial weight diagram. Each spin state should have the same energy (if no additional dynamics are inforced, like external magnetic fields). If we say that physics should be invariant under an interchange of u,d,s quarks, called $SU(3)_{flavour}$ symmetry, then the non trivial weight diagrams can explain why certain hadrons containing these quarks have similar properties. For example, as mentioned, each basis element in the representation space for a particular representation should have similar properties like energy. The representation space for $SU(3)_{flavour}$ is most easily pictured in Cartan space, where we see that the weight diagrams discribe which particles should have similar properties, in for example the 8-fold way. When we only consider the subgroup of $SU(3)_{flavour}$, where only an interchange of u,d quarks is a symmetry, this is called $SU(2)_{isopsin}$. Flavour symmetry is broken by the fact that the quarks have different masses and can have different electric charges. However, it is still an approx. symmetry of nature.

Now back to the discussion at hand. What is happening with the Cartan alegbra and weights? Consider plain old and simple quantum mechanics! We do what we always do. We find a maximally set of commuting observables, O_i. Since they are all commuting, we can simultaneously diagonalise them (this is why we only consider commuting operators). When we diagonalise the operators, we can then find a basis for our states which are eigenvectors of all the operators O_i. Label this basis, $|\psi\rangle^k$, e.g, we have $O_i|\psi\rangle^k=m_i^k|\psi\rangle^k$.

[1] See section 8.9 here if interested: http://cft.fis.uc.pt/eef/evbgroups.pdf

As a reminder, it is these eigenvalues that we can simultaneously measure in experiment, which is why we are actually interested in them. We can now associate to each $|\psi\rangle^k$ a set of eigenvalues $\{m_1^k, m_2^k, \ldots, m_r^k\}$, where r is the number of communting operators. What can we do with this set of eigenvalues? Well, since they are the interesting quantities we are interested in, we can relabel the state by what we can observe: $|\psi\rangle^k = |m_1^k, \ldots, m_r^k\rangle^k$[2]. Then a true state of our theory is a superposition of all these basis vectors. Let's take a simple example, $SU(2)_{spin}$. Say our set of maximally commuting operators consists of S^2, S_z. We diagonalise these operators, and find a basis $|\psi\rangle^m$ which are eigenvectors of these operators. Eigenvalues of these operators are labeled by s for S^2, and m for S_z, where $m \in -s, \ldots, s$. We can therefore label the states as $|\psi\rangle^m = |s, m\rangle$, and we can measure this spin of a state experimentially if we so desire.

So how does something like this generalise to other groups, and what information can we get out of it? The argument generalises straightforwardly to other symmetry groups. The information we get is most easily shown in terms of a weight diagram. Again, consider the maximally set of commuting generators of the Lie algebra of a symmetry (exponentiating each of these generators will produce a basis element of the covering group, which will commute with the other exponentiated elements by the BCH theorem). We can simultaneously diagonalise these generators and find a basis which are eigenvectors of these generators. Call these generators, in a particular irrep, by $d(H_i)$, and a basis for the representation space by $|\psi\rangle^j$. Let there be r generators in the Cartan algebra (so that $i \in 1, \ldots, r$) and the representation be N-dimensional (so that $j \in 1, \ldots, N$). Identical to the previous discussion, $d(H_i)|\psi\rangle^j = m_i^j|\psi\rangle^j$. We can now associate to every basis element a set of eigenvalues $\{m_1^j, m_2^j, \ldots, m_r^j\}$. If we wanted, we could again relabel the basis of the representation space as $|\psi\rangle^j = |m_1^j, m_2^j, \ldots, m_r^j\rangle^j$. However, this doesn't easily communicate all the physically interesting information about the states in our theory. To easily describe the different states of our theory, it is useful to go to Cartan space and draw a weight diagram. To do this, for every basis vector of our representation space $|\psi\rangle^j = |m_1^j, m_2^j, \ldots, m_r^j\rangle^j$, we associate a vector in Cartan space, $(m_1^j, m_2^j, \ldots, m_r^j)$, called a weight vector. When we plot these different weight vectors in Cartan space, they build up a picture of how the states are grouped for a particular representation. This was historically important as the famous 8-fold way of $SU(3)_{flavour}$ explains why many hadrons have similar properties. Note that the symmetry group that we choose will place strict constraints on which weight vectors are allowed for a particular irrep, and hence give different looking weight diagrams. For example, compare the three dimensional weight diagram of $SU(2)$ to that of $SU(3)$.

One might be wondering how to build up a weight diagram without doing all the work of classifying all the eigenvalues of the Cartan algebra. Well, one can decompose the Cartan algebra into certain combinations of generators, so that these generators look like a number of $SU(2)$ Lie algebra's[3]. This allows us to draw the weight diagram of the symmetry group as combinations of $SU(2)$ weight diagrams (which are just straight lines). Roots are defined to be the weight vectors in the adjoint representation and they are important as they tell us which direction we can move in weight diagram. So using just one weight and the roots, we can build up a full weight diagram for a particular represention.

Question 1

This question is basically bookwork, you can find it in for instance the notes of Hugh Osborn or Jan Gutowski for older versions of this course. You should also have seen it in quantum mechanics. In any case, here is a solution (with thanks to Arnab Rudra for providing it).

Let V_j be the irreducible representation with highest weight j. The states in V_j can be denoted $|j, j-m\rangle$ for $m = 0, \ldots, 2j$, so $\dim V_j = 2j+1$, where the weight of $|j, j-m\rangle$ is $j-m$. Now take $j \geq j'$ and consider $V_j \otimes V_{j'}$. The J_3 operator in the tensor product representations will be $J_3 \otimes I' + I \otimes J_3'$ (see next question) so that weights add.

Therefore the weights of the tensor product are given by the sums of the weights of V_j and V_j' and so are of the form $M = j + j' - k$. Let us denote by U_M the vector subspace of all states with weight M, and concentrate just on the states with $M \geq 0$ (which are our candidate states to become highest weight states of irreducible representations). Suppose first that $k \leq 2j'$, i.e. $M \geq j - j'$. Then in this case the multiplicity of states with weight M is given by the number of ways you can write M as

$$M = (j-k) + j' = (j-(k-1)) + (j'-1) = \cdots = j + (j'-k)\,. \tag{1.1}$$

So the multiplicity of the state with weight M for $M \geq j - j'$ is $k+1$. We have $\dim U_M = k+1 = j + j' - M + 1$.

[2]The combinations of eigenvalues allowed is dictated by the symmetry that you choose (or for experimental physics, that nature obeys).
[3]These $SU(2)$ Lie algebra's may not all commute with one another

For $M < j - j'$, i.e. $k > 2j'$ then $\dim U_M = 2j' + 1$, as

$$M = (j-k) + j' = (j-(k-1)) + (j'-1) = \cdots = (j-(k-2j')) + (j'-2j') \,. \tag{1.2}$$

Now let's try and find the decomposition into irreducible factors. Assume $M \geq j - j'$. Consider U_{M+1} and note that as J_- lowers the weight by one we have $J_- U_{M+1} \subset U_M$. However $\dim U_{M+1} = j + j' - (M+1) + 1$ and $\dim U_M = j + j' - M + 1$, so the dimension of U_M is one greater than U_{M+1}. So we must be able to find a vector $|\phi\rangle$ in U_M which is orthogonal to all elements in $J_- U_{M+1}$. We claim that $J_+|\phi\rangle = 0$, meaning we can take $|\phi\rangle$ as the highest weight vector of some irreducible representation.

To prove this claim, suppose $J_+|\phi\rangle \neq 0$, so that $J_+|\phi\rangle = |\Phi\rangle \in U_{M+1}$. Then

$$\langle\Phi|\Phi\rangle = \langle\phi|J_- J_+|\phi\rangle = \langle\phi|(J_-|\Phi\rangle) = 0 \tag{1.3}$$

using in the last step the fact that $|\phi$ is orthogonal to all elements in $J_- U_{M+1}$ and the fact that $|\Phi\rangle \in U_{M+1}$. Hence we get a set of highest weight states, starting with $|j+j', j+j'\rangle$ and continuing down to $|j-j', j-j'\rangle$.

Note that for $M < j - j'$ the above argument breaks down as then U_{M+1} and U_M both have dimension $2j' + 1$. So we don't get any more irreducible representations.

We thus have a decomposition

$$V_j \otimes V_{j'} = V_{j+j'} \oplus V_{j+j'-1} \oplus \cdots \oplus V_{j-j'} \,. \tag{1.4}$$

The dimension of the left-hand side is $(2j+1)(2j'+1)$. That of the right-hand side is

$$\sum_{k=0}^{2j'} (2(j+j'-k)+1) = (2(j+j')+1)\sum_{k=0}^{2j'} 1 - 2\sum_{k=1}^{2j'} k = (2j'+1)(2j+2j'+1) - 2j'(2j'+1) = (2j+1)(2j'+1) \tag{1.5}$$

Here we used

$$\begin{aligned} \sum_{k=1}^{n} k = {} & 1+ \\ & 1+1+ \\ & 1+1+1+ \\ & \vdots \\ & \underbrace{1+1+1+\cdots+1}_{n} \\ & = \frac{1}{2}n(n+1) \end{aligned} \tag{1.6}$$

i.e. the number of 1s appearing here is equal to the number of components of an $n \times n$ symmetric matrix.

Question 2

a) Consider

$$\mathcal{D}(ghg^{-1}) = \mathcal{D}(g)\mathcal{D}(h)\mathcal{D}(g)^{-1} \tag{2.1}$$

where $h \in G$. We now expand $h = I + X$ for X in the Lie algebra, then

$$\mathcal{D}(I + gXg^{-1}) = \mathcal{D}(g)\mathcal{D}(I+X)\mathcal{D}(g)^{-1} \tag{2.2}$$

and we can then expand the lhs in terms of Lie algebra representations

$$I + d(gXg^{-1}) = I + \mathcal{D}(g)d(X)\mathcal{D}(g)^{-1} \tag{2.3}$$

giving the result.

b) Let $\mathcal{D}^{(1)}(g) = I + td^{(1)}(X) + O(t^2)$, $\mathcal{D}^{(2)}(g) = I + td^{(2)}(X) + O(t^2)$, then

$$\begin{aligned} \mathcal{D}^{(1)}(g) \otimes \mathcal{D}^{(2)}(g) &= (I + td^{(1)}(X)) \otimes (I + td^{(2)}(X)) + O(t^2) \\ &= I + t\left(d^{(1)}(X) \otimes I + I \otimes d^{(2)}(X)\right) + O(t^2)\,. \end{aligned} \tag{2.4}$$

Hence the associated representation of the Lie algebra is indeed

$$d(X) = d^{(1)}(X) \otimes I + I \otimes d^{(2)}(X)\,. \tag{2.5}$$

Note that the two identity factors are acting on different spaces. In matrix index notation we have

$$d(X)_{ab\alpha\beta} = d^{(1)}(X)_{ab}\delta_{\alpha\beta} + \delta_{ab}d^{(2)}(X)_{\alpha\beta}\,. \tag{2.6}$$

Let $|\lambda\rangle$ be an eigenvector of $d^{(1)}(X)$ with eigenvalue λ and let $|\mu\rangle$ be an eigenvector of $d^{(2)}(X)$ with eigenvalue μ. Then

$$d(X)|\lambda\rangle \otimes |\mu\rangle = \left(d^{(1)}(X) \otimes I + I \otimes d^{(2)}(X)\right)|\lambda\rangle \otimes |\mu\rangle = (\lambda + \mu)|\lambda\rangle \otimes |\mu\rangle \tag{2.7}$$

so that $|\lambda\rangle \otimes |\mu\rangle$ is an eigenvector of $d(X)$ with eigenvalue $\lambda + \mu$. Therefore the sums of all pairs of eigenvalues of $d^{(1)}(X)$ and $d^{(2)}(X)$ are eigenvalues of $d(X)$. In general the weights of a representation are defined to be the eigenvalues of some set of mutually commuting generators $d(H_i)$, this result shows that the weights of the tensor product representation are given by all possible sums of weights of the individual representations in the tensor product.

Question 3

The scalar field Φ transforms as $\Phi \to \mathcal{D}(g)\Phi$ and the gauge potential A_μ, which lives in the Lie algebra, transforms as $A_\mu \to gA_\mu g^{-1} - \partial_\mu g g^{-1}$.

We define the covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi + d(A_\mu)\Phi\,. \tag{3.1}$$

Seeing as $A_\mu \in L(G)$ transforms under $g \in G$ it follows (from question 2) that $d(A_\mu) \in L(\mathcal{D}(G))$ transforms under $\mathcal{D}(g) \in \mathcal{D}(G)$, and in total we have

$$D_\mu\Phi \to \partial_\mu(\mathcal{D}(g)\Phi) + \left(\mathcal{D}(g)A_\mu\mathcal{D}(g)^{-1} - \partial_\mu\mathcal{D}(g)\mathcal{D}(g)^{-1}\right)\mathcal{D}(g)\Phi = \mathcal{D}(g)D_\mu\Phi\,. \tag{3.2}$$

To construct a term in the Lagrangian density from $D_\mu\Phi$ we need some quadratic form $(D_\mu\Phi, D^\mu\Phi)$ which is invariant under $D_\mu\Phi \to \mathcal{D}(g)D_\mu\Phi$, i.e. $(\mathcal{D}(g)D_\mu\Phi, \mathcal{D}(g)D^\mu\Phi) = (D_\mu\Phi, D^\mu\Phi)$.

Another way of putting this is to say we are in general looking for some expression $\overline{D_\mu\Phi}D^\mu\Phi$. What we mean by $\overline{\Phi}$ depends on the properties of the group and representation. For example, if $\mathcal{D}$ is a unitary representation (which will be true if G is compact) then the natural kinetic term is $(D_\mu\Phi)^\dagger D^\mu\Phi$. In other cases $\overline{\Phi}$ could mean just the transpose, if Φ is real, or something more complicated involving contraction with an invariant tensor which defines a quadratic form as above.

In slightly more formal terms if we denote by $\overline{\mathcal{D}}$ the representation conjugate to $\mathcal{D}$ then we need the tensor product $\mathcal{D} \otimes \overline{\mathcal{D}}$ to contain the trivial representation. This would then guarantee the existence of an invariant formed from $\overline{D_\mu\Phi}$ and $D^\mu\Phi$ which could potentially be used as a kinetic term in a Lagrangian.

Question 4

We have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{4.1}$$

The gauge transform of A_μ is

$$A_\mu \to gA_\mu g^{-1} - \partial_\mu g g^{-1}\,. \tag{4.2}$$

Note that

$$0 = \partial_\mu(gg^{-1}) \Rightarrow \partial_\mu g g^{-1} = -g\partial_\mu g^{-1} \,, \tag{4.3}$$

which we will use repeatedly below. We're asked to directly determine how $F_{\mu\nu}$ transforms. This means

$$\begin{aligned}
F_{\mu\nu} &\to \partial_\mu(gA_\nu g^{-1} - \partial_\nu g g^{-1}) + (gA_\mu g^{-1} - \partial_\mu g g^{-1})(gA_\nu g^{-1} - \partial_\nu g g^{-1}) - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu g)A_\nu g^{-1} + gA_\nu\partial_\mu g^{-1} + g(\partial_\mu A_\nu)g^{-1} - \partial_\nu g\partial_\mu g^{-1} + gA_\mu A_\nu g^{-1} - gA_\mu g^{-1}\partial_\nu g g^{-1} - \partial_\mu g A_\nu g^{-1} + \partial_\mu g g^{-1}\partial_\nu g g^{-1} \\
&\qquad - (\mu \leftrightarrow \nu) \\
&= g(\partial_\mu A_\nu + A_\mu A_\nu)g^{-1} + gA_\nu\partial_\mu g^{-1} + gA_\mu\partial_\nu g^{-1} - \partial_\nu g\partial_\mu g^{-1} - \partial_\mu g\partial_\nu g^{-1} - (\mu \leftrightarrow \nu) \\
&= gF_{\mu\nu}g^{-1} \,.
\end{aligned} \tag{4.4}$$

To do the last part of the question, one can explicit plug $A_\mu = -\partial_\mu g g^{-1}$ into the expression for $F_{\mu\nu}$ and show that the latter vanishes. To save ourselves the effort we can instead note that in this case one really has $A_\mu = g0_\mu g^{-1} - \partial_\mu g g^{-1}$, i.e. A_μ is a gauge transformation of the zero (flat) connection. The field strength of the zero connection is obviously zero, so the field strength associated to this A_μ is $F_{\mu\nu} = g0_{\mu\nu}g^{-1} = 0$.

Question 5

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^* D^\mu\phi - \frac{1}{4}\lambda\left(\phi^*\phi - v^2\right)^2 \,. \tag{5.1}$$

Here $D_\mu\phi = \partial_\mu\phi - ia_\mu\phi$. We let $\phi = e^{i\beta}(v + \eta(x))$, so that $\partial_\mu\phi = i\phi\partial_\mu\beta + e^{i\beta}\partial_\mu\eta$. Then

$$\begin{aligned}
(D_\mu\phi)^* D^\mu\phi &= \left(-i(\partial_\mu\beta - a_\mu)\phi^* + e^{-i\beta}\partial_\mu\eta\right)\left(i(\partial^\mu\beta - a^\mu)\phi + e^{i\beta}\partial^\mu\eta\right) \\
&= (v+\eta)^2(\partial_\mu\beta - a_\mu)(\partial^\mu\beta - a^\mu) + \partial_\mu\eta\partial^\mu\eta + i(\partial_\mu\beta - a_\mu)\partial^\mu\eta\phi e^{-i\beta} - i(\partial_\mu\beta - a_\mu)\partial^\mu\eta\phi^* e^{i\beta}
\end{aligned} \tag{5.2}$$

and the last two terms clearly cancel. The scalar potential becomes

$$\frac{1}{4}\lambda\left(\phi^*\phi - v^2\right)^2 = \frac{1}{4}\lambda\left((v+\eta)^2 - v^2\right)^2 = \frac{\lambda}{4}\eta^2\left(\eta + 2v\right)^2 \,. \tag{5.3}$$

Hence we can write

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}(v+\eta)^2(a_\mu - \partial_\mu\beta)(a^\mu - \partial^\mu\beta) - \frac{\lambda}{4}\eta^2\left(\eta + 2v\right)^2 \,. \tag{5.4}$$

We observe that there is now a mass term for η:

$$-\frac{1}{2}m_\eta^2\eta^2 \equiv -\lambda v\eta^2 \Rightarrow m_\eta^2 = 2\lambda v \tag{5.5}$$

and also one for the gauge field:

$$\frac{1}{2}m_a^2 a_\mu a^\mu \equiv \frac{1}{2}v^2 a_\mu a^\mu \Rightarrow m_a^2 = v^2 \,. \tag{5.6}$$

You might be worried that this term appears in Lagrangian with a different sign to the scalar mass term, however because we are using the mostly minus metric (as is apparent from the fact the kinetic terms $+\partial_\mu\eta\partial^\mu\eta = +\partial_0\eta\partial^0\eta + \dots$) for the spatial components this means $+\frac{1}{2}m_a^2 a_\mu a^\mu \to -\frac{1}{2}m^2 a_i a_i$, which is the right sign.

Now, the gauge transformations of the original Lagrangian were

$$\phi \to e^{i\lambda}\phi \,, \; a_\mu \to a_\mu + \partial_\mu\lambda \tag{5.7}$$

(in this question, unlike the previous one, we are using the physical definition of gauge potentials, where a_μ is real and there is an extra factor of i. We can transform back to the previous notation by letting $A_\mu = -ia_\mu$, so that the transformation rule $A_\mu \to gA_\mu g^{-1} - \partial_\mu g g^{-1} = -i(ga_\mu g^{-1} - i\partial_\mu g g^{-1})$, so $a_\mu \to ga_\mu g^{-1} - i\partial_\mu g g^{-1}$. The U(1) gauge transformation $g = e^{i\lambda}$ then gives the above simple transformation of a_μ). It's clear that these transformations act on β and η as

$$\beta \to \beta + \lambda\ , \ \eta \to \eta\,. \tag{5.8}$$

The transformed Lagrangian is still gauge invariant because a_μ and β occur together in the gauge invariant quantity $a_\mu - \partial_\mu\beta$.

We could define a new gauge field $\tilde{a}_\mu = a_\mu - \partial_\mu\beta$ and eliminate β from the Lagrangian altogether, leaving us with a theory of a massive vector field and a massive scalar. Note that in four dimensions a massless vector field has two physical degrees of freedom (corresponding to its two possible helicities) which are so-called transverse degrees of freedom, while a massive vector has three physical degrees of freedom, with the extra one relative to the massless case being the longitudinal degree of freedom.

In our abelian Higgs model we originally had a massless vector field, with its two degrees of freedom, and a complex scalar, which also has two (real) physical degrees of freedom. After "Higgsing", we have a massive vector field which has three independent physical degrees of freedom, and a real scalar with one physical degree of freedom. We see that the degree of freedom of ϕ represented by β has been "eaten" by the massless vector field to turn itself into a massive vector - the new vector degree of freedom corresponds to longitudinal excitations.

Question 6

The SU(2) subgroup of U(2) is just found by restricting to matrices of determinant one in U(2), see example sheet 1. A U(1) subgroup consists of diagonal matrices of the form

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{6.1}$$

which are certainly unitary and commute with everything else in the group, in particular the SU(2) subgroup.

Let U be any element of U(2) and let $\det U = e^{i\alpha}$. Then define $U' = e^{-i\alpha/2}U$, which clearly has unit determinant and so is an SU(2) matrix. It follows trivially then that any element U of U(2) can be written in the form $U'e^{i\alpha/2}I$, where $U' \in$ SU(2) and $e^{i\alpha/2}I \in$ U(1). Hence the homomorphism from SU(2) $\times$ U(1) to U(2) is onto.

If $U = I$ then we have $U'e^{i\alpha/2} = I$. Remembering the general decomposition of an SU(2) matrix this means the kernel consists of matrices such that

$$I = \begin{pmatrix} e^{i\theta}e^{i\alpha/2} & 0 \\ 0 & e^{-i\theta}e^{i\alpha/2} \end{pmatrix} \tag{6.2}$$

which is only possible in the cases that $\theta = \alpha = 0$ or $\theta = \pi$, $\alpha = 2\pi$, and hence the kernel consists of the pairs $(+I, 1)$ and $(-I, -1)$. The first of these is the identity, and the second squares to it, so this gives a group isomorphic to $\mathbb{Z}_2$. The fundamental theorem on homomorphisms then implies $\mathrm{U}(2) = (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$.

The analogous result for U(n) is $\mathrm{U}(n) = (\mathrm{SU}(n) \times \mathrm{U}(1))/\mathbb{Z}_n$. This can be seen by first noting we can use the same trick with the determinant to write any element of U(n) in terms of a U(1) factor and a special unitary matrix. Then the general element giving the identity has the form

$$e^{i\phi}\mathrm{diag}(a_1, \ldots a_n) \tag{6.3}$$

where we must have $a_1 \ldots a_n = 1$ so that we have an SU(2) matrix. Now, we separately need

$$a_1 e^{i\phi} = 1\ a_2 e^{i\phi} = 1\ \ldots\ a_n e^{i\phi} = 1\,. \tag{6.4}$$

However as $a_n = a_1^{-1} \ldots a_{n-1}^{-1}$ and $a_i^{-1} = e^{i\phi}$ this means we have $e^{in\phi} = 1$, i.e. $e^{i\phi}$ must be an nth root of unity. So the kernel consists of pairs

$$(\omega I, \omega^{-1}) \tag{6.5}$$

where $\omega^n = 1$, which generates a $\mathbb{Z}_n$.

Now consider Φ transforming as $\Phi \to U\Phi$. Take $\Phi_0 = (v, 0)^T$. Using our decomposition of U(2) into U(1) and SU(2) factors we have

$$\Phi_0 \to e^{i\phi/2} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = e^{i\phi/2} \begin{pmatrix} v\alpha \\ -\beta^* v \end{pmatrix} . \tag{6.6}$$

The orbit of Φ_0 then consists of points $(x, y)^T$ with $|x|^2 + |y|^2 = v^2(|\alpha|^2 + |\beta|^2) = v^2$, using that $|\alpha|^2 + |\beta|^2 = 1$ as usual for an SU(2) matrix. What this tells us that the points on the orbits of Φ_0 are constrained to lie on a three-sphere (of radius v). So $M = S^3$.

The isotropy group of Φ_0 consists of all transformations that don't actually transform it. We see from the above this means $\beta = 0$, which implies that $\alpha = e^{i\theta}$ is a unit complex number, and for Φ_0 to be unchanged we need $e^{i(\theta+\phi/2)} = 1$. So we should take $\theta = -\phi/2$, in which case the group element has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{6.7}$$

which generates a U(1) subgroup.

As a homogenous space then $M = \text{U}(2)/\text{U}(1)$ (we should also check that the action is transitive, i.e. we get from any element to any other, but this is more or less obvious as we can get from Φ_0 to any element of the sphere (choose $\phi = 0$ and use that S^3 is isomorphic to SU(2)), and can invert the transformation to go from that element back to Φ_0 and on to anything else).

Another way of writing S^3 as a homogeneous space is as SO(4)/SO(3). This is not hard to see, let's consider the more general case $S^{n-1} = \text{SO}(n)/\text{SO}(n-1)$. Recall from a previous example sheet that an element of O(n) (and hence SO(n) too) can be written in terms of n orthonormal column vectors. So if we consider the action of SO(n) on n-dimensional vectors and choose $v_0 = (1, 0, \dots, 0)^T$ then under a transformation $U = (u_1, \dots, u_n)$ we have $v_0 \to u_1$. As u_1 is an n-component unit vector we see that the orbit of v_0 is an $(n-1)$-sphere. It's also easy to see that the isometry group leaving v_0 fixed consists of matrices with 1 in the very first entry and an SO($n-1$) block in the remaining rows and columns, so that indeed $S^{n-1} = \text{SO}(n)/\text{SO}(n-1)$.

Indeed, we can also generalise $S^{2n-1} = \text{U}(n)/\text{U}(n-1)$ to obtain more examples (the first column of a U(n) will be an n-component complex vector obeying $u^\dagger u = 1$, etc).

Question 7

Suppose that $A\mathcal{D}(g) = \mathcal{D}(g)A$ for all g in G. Let W be an eigenspace in the representation space V carrying the irreducible representation $\mathcal{D}$, with eigenvalue λ, i.e. $Aw = \lambda w$ for all $w \in W$. Then for $w \in W$ we have $A\mathcal{D}(g)w = \mathcal{D}(g)Aw = \lambda\mathcal{D}(g)w$. So $\mathcal{D}(g)w$ is also an eigenvector of A with eigenvalue λ, and so lies in W. Hence we have found an invariant subspace W of V. But we assumed that the representation we are considering is irreducible, hence V cannot have any non-trivial invariant proper subspaces. Thus either $\lambda = 0$ and W is the trivial subspace, or else W must in fact be V itself, with A acting as λI_V.

Question 8

Let $Q_{ij} = \text{tr}\,(d(T_i)d(T_j))$. The trace is over the representation d, and we can think of Q as defining a matrix which acts on the adjoint representation (i.e. on the Lie algebra itself). Recall the generators in the adjoint are given by $\text{ad}(T_i)_{jk} = -f_{ijk}$ (with totally antisymmetric structure constants as the Lie algebra is assumed to be of compact type). Following the hint we consider

$$(Q\text{ad}(T_i))_{jl} = -Q_{jk} f_{ikl} \tag{8.1}$$

and

$$(\text{ad}(T_i)Q)_{jl} = -f_{ijk} Q_{lk} \tag{8.2}$$

(note that Q is symmetric by the properties of the trace). Now, observe that

$$-Q_{jk} f_{ikl} = f_{ilk} Q_{jk} = \text{tr}\,([d(T_i), d(T_l)]d(T_j)) , \tag{8.3}$$

and

$$-f_{ijk} Q_{lk} = f_{jik} Q_{lk} = \text{tr}\,([d(T_j), d(T_i)]d(T_l)) . \tag{8.4}$$

However,

$$\operatorname{tr}\left([d(T_j), d(T_i)]d(T_l)\right) = \operatorname{tr}\left(d(T_j)d(T_i)d(T_l) - d(T_i)d(T_j)d(T_l)\right) = \operatorname{tr}\left(d(T_i)d(T_l)d(T_j) - d(T_l)d(T_i)d(T_j)\right) \tag{8.5}$$

so in fact $Q\mathrm{ad}(T_i) = \mathrm{ad}(T_i)Q$, i.e. Q commutes with the action of the adjoint representation. Applying Schur's lemma (which we proved for Lie groups but holds similarly for representations of Lie algebras) we conclude that if the adjoint representation is irreducible that Q is proportional to the identity operator. As in this question we're considering a simple Lie algebra the adjoint is indeed irreducible - simple means that there are no invariant ideals of the algebra, i.e. subspaces closed under the action of the whole algebra by commutation, which is just the adjoint action. So therefore $Q_{ij} = \lambda\delta_{ij}$.

We can lastly fix the sign of λ as follows. As we're told the Lie algebra is of compact type, then the corresponding compact group has unitary representations, which means those of the algebra must be anti-hermitian[4], $d(T_i)^\dagger = -d(T_i)$. So $Q_{ii} = \operatorname{tr}(d(T_i)d(T_i)) = -\operatorname{tr}(d(T_i)^\dagger d(T_i))$ (no sum on i). However $\operatorname{tr}(d(T_i)^\dagger d(T_i))$ defines a positive definite form so therefore Q_{ij} is negative definite, i.e. $Q_{ij} = -\lambda\delta_{ij}$ with $\lambda > 0$.

Question 9

Consider the action

$$S = -\int dt \operatorname{tr}\left(\dot{g}g^{-1}\dot{g}g^{-1}\right) = -\int dt \operatorname{tr}\left(g^{-1}\dot{g}g^{-1}\dot{g}\right) . \tag{9.1}$$

Let's say we vary the trajectory $g(t) \to g(t) + \delta g(t)$, where $\delta g(t)$ is infinitesimal. We can express

$$g(t) + \delta g(t) = g(t)(I + g^{-1}\delta g(t)) . \tag{9.2}$$

Now the quantity in brackets is really an infinitesimal variation close to the identity so that $g^{-1}\delta g(t)$ must then be an element $\delta X(t)$ of the Lie algebra. Hence $g(t) + \delta g(t) = g(t) + g(t)\delta X(t)$, i.e. the variation is $\delta g(t) = g(t)\delta X(t)$.

Let's see what happens when we vary the action. We have

$$\delta S = -2\int dt \operatorname{tr}\left(\delta(g^{-1}\dot{g})g^{-1}\dot{g}\right) . \tag{9.3}$$

Now, $\delta(gg^{-1}) = 0$ implies as usual that $\delta g^{-1} = -g^{-1}\delta g g^{-1}$, so

$$\delta S = -2\int dt \operatorname{tr}\left(-g^{-1}\delta g g^{-1}\dot{g}g^{-1}\dot{g} + g^{-1}\delta\dot{g}g^{-1}\dot{g}\right) , \tag{9.4}$$

and we can also write

$$\begin{aligned} g^{-1}\delta\dot{g}g^{-1}\dot{g} &= \frac{d}{dt}\left(g^{-1}\delta g g^{-1}\dot{g}\right) - \left(\frac{d}{dt}g^{-1}\right)\delta g g^{-1}\dot{g} - g^{-1}\delta g\frac{d}{dt}\left(g^{-1}\dot{g}\right) \\ &= \frac{d}{dt}\left(g^{-1}\delta g g^{-1}\dot{g}\right) + g^{-1}\dot{g}g^{-1}\delta g g^{-1}\dot{g} - g^{-1}\delta g\frac{d}{dt}\left(g^{-1}\dot{g}\right) , \end{aligned} \tag{9.5}$$

and so, throwing away the boundary terms and using the cyclic property of the trace, we end up with

$$\delta S = 2\int dt \operatorname{tr}\left(g^{-1}\delta g\frac{d}{dt}\left(g^{-1}\dot{g}\right)\right) . \tag{9.6}$$

Note that if we had started with the original form of the action we would have gotten

$$\delta S = 2\int dt \operatorname{tr}\left(\delta g g^{-1}\frac{d}{dt}\left(\dot{g}g^{-1}\right)\right) . \tag{9.7}$$

[4] More accurately, all the group representations are equivalent to unitary representations by change of basis, and all the algebra representations are equivalent to antihermitian representations by change of basis. Seeing as Q involves a trace it is invariant under change of basis so our results hold in general. Note that if $\mathcal{D}(g)^\dagger\mathcal{D}(g) = 1$ and $\mathcal{D}(g) = I + d(g)$ then to linear order we do indeed require $d(g) = -d(g)^\dagger$.

From (9.6) we see that if $\delta g = g\delta X$ then for arbitrary variations[5] δX we must have

$$\frac{d}{dt}\left(g^{-1}\dot{g}\right) = 0\,. \tag{9.8}$$

Now, differentiating in we have

$$g^{-1}\ddot{g} - g^{-1}\dot{g}g^{-1}\dot{g} = 0 \Rightarrow \ddot{g} = \dot{g}g^{-1}\dot{g}\,. \tag{9.9}$$

We consider then

$$\frac{d}{dt}\left(\dot{g}g^{-1}\right) = \ddot{g}g^{-1} - \dot{g}g^{-1}\dot{g}g^{-1}\,, \tag{9.10}$$

and this vanishes when we sub in for $\ddot{g}$ as above, so both these quantities are "time" independent (alternatively could have used eom following from variation (9.7), noting that $g\delta X g^{-1}$ is also an arbitrary variation in the Lie algebra [or slightly more directly, could have pulled out a factor of g from the right instead of the left in (9.2) to have variations $\delta g = \delta Y g$]).

Now consider $g(t) = g_0 \exp(tX_0)$, and suppose that we have chosen the time coordinate such that the trajectory starts at $t = 0$ when $g = g_0$. Using the time independence we know that we can simply evaluate $g^{-1}\dot{g}$ and $\dot{g}g^{-1}$ at any particular time of our choosing. The most convenient time is clearly $t = 0$, as

$$g(t) = g_0\left(I + tX_0 + \frac{t^2}{2}X_0^2 + \ldots\right) \Rightarrow \dot{g}(t)\Big|_{t=0} = g_0 X_0 \tag{9.11}$$

so then simply

$$g^{-1}\dot{g} = X_0\ ,\ \dot{g}g^{-1} = g_0 X_0 g_0^{-1}\,. \tag{9.12}$$

Question 10

We consider the matrix

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{10.1}$$

and define SU(1, 1) to be the set of unit determinant 2 by 2 matrices U such that $U^\dagger \eta U = \eta$. To find the Lie algebra we look at elements close to the identity, i.e. we let $U = I + A$ with A considered infinitesimal, so that $(I + A)^\dagger \eta (I + A) = I$ implies (to order A) that $A^\dagger \eta + \eta A = 0$. We also know A must be traceless in order that U has unit determinant, so we can in general write

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\,, \tag{10.2}$$

so that

$$A^\dagger \eta + \eta A = \begin{pmatrix} a^* & c^* \\ b^* & -a^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a + a^* & b - c^* \\ b^* - c & a + a^* \end{pmatrix}\,. \tag{10.3}$$

We see that we need $a = -a^*$, i.e. a is pure imaginary, and $c = b^*$. Thus the Lie algebra consists of matrices of the form

$$\begin{pmatrix} ia_3 & a_1 + ia_2 \\ a_1 - ia_2 & -ia_3 \end{pmatrix}\,, \tag{10.4}$$

[5] One can note here that as $g^{-1}\delta g$ is an element of the Lie algebra, and the group is assumed compact, then this trace is indeed an inner product by arguments about (anti-)hermicity as in the previous question. Both δX and $g^{-1}\dot{g}$ lie in the Lie algebra.

where the a_i are real numbers. We select the "standard basis"

$$\begin{aligned} T_1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1 \,, \\ T_2 &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_2 \,, \\ T_3 &= \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{i}{2}\sigma_3 \,. \end{aligned} \tag{10.5}$$

The commutation relations are then

$$[T_1, T_2] = T_3 \ , \ [T_3, T_1] = -T_2 \ , \ [T_2, T_3] = -T_1 \,. \tag{10.6}$$

This means the structure constants (defined here by $[T_a, T_b] = f_{abc}T_c$) are

$$f_{123} = 1 = -f_{213} \ , \ f_{312} = -1 = -f_{132} \ , \ f_{231} = -1 = -f_{321} \ , \tag{10.7}$$

The Killing form κ_{ab} is defined by $\kappa_{ab} = \operatorname{tr}(\operatorname{ad}(T_a)\operatorname{ad}(T_b))$. Now, we know that $[\operatorname{ad}(T_a)]_{bc} = f_{acb}$ so that the definition of the Killing form then gives $\kappa_{ab} = f_{acd}f_{bdc}$. In our case it's clear that we need a and b to be equal to get a non-zero value. Then we have

$$\kappa_{11} = f_{1cd}f_{1dc} = f_{123}f_{132} + f_{132}f_{123} = 2 \tag{10.8}$$

$$\kappa_{22} = f_{2cd}f_{2dc} = f_{231}f_{213} + f_{213}f_{231} = 2 \tag{10.9}$$

$$\kappa_{33} = f_{3cd}f_{3dc} = f_{312}f_{321} + f_{321}f_{312} = -2 \tag{10.10}$$

so that the Killing form is

$$\kappa_{ab} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{10.11}$$

This is indeed non-degenerate (invertible) and is not of compact type as it is not negative definite - the quadratic form it defines is $2(x^2 + y^2 - z^2)$ which is not everywhere negative.

To find a compact subgroup of SU(1, 1) we will exponentiate some carefully chosen generator of the Lie algebra. One way to do this is to guess in an educated manner - the generator T_3 squares to minus the identity (up to a positive numerical factor). So when we exponentiate it we expect to get the familiar decomposition into sine and cosine terms, these functions being bounded implying we have something compact. Conversely T_1 and T_2 square to something proportional to the identity, and so will give cosh and sinh when exponentiated, which are unbounded functions. This should make more sense when you actually do the calculations (results are below).

More rigourously we can use the result that negative definite Killing form implies compactness. So we can look at the Killing form and restrict to subspaces on which it is positive or negative definite to identity Lie algebra subspaces of compact or non-compact type. As $\kappa_{33} < 0$ restricting to just T_3 gives us a compact subspace, while $\kappa_{11}, \kappa_{22} > 0$ implies T_1, T_2 generate a non-compact subspace.

So to explicitly obtain a compact subgroup we exponentiate the generator T_3: we have

$$\exp(\phi T_3) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} . \tag{10.12}$$

The parameter ϕ here runs from 0 to 2π giving us a U(1) subgroup. A non-compact subgroup can be found by exponentiating one of the other generators, for instance as $(2T_1)^2 = I$ we have

$$\exp(2\alpha T_1) = \cosh\alpha + 2T_1 \sinh\alpha = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} . \tag{10.13}$$

The parameter α here ranges over all of $\mathbb{R}$, so this gives a subgroup $\mathbb{R}$ in $\mathrm{SU}(1,1)$.

Part III Symmetries, Fields and Particles (Michaelmas 2013): Example Sheet 4 Solutions

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Thanks to Alex Arvanitakis and Arnab Rudra for various corrections and suggestions.

Please send me comments and corrections.

Question 1

We have $\kappa_{ij}=\kappa(T_i,T_j)=\mathrm{tr}(\mathrm{ad}T_i\mathrm{ad}T_j)$. Note that $\kappa_{ij}=\kappa_{ji}$. The hint suggests we consider

$$\kappa(T_i,[T_j,T_k])=\kappa([T_i,T_j],T_k)\Rightarrow c_{jkl}\kappa_{il}=c_{ijl}\kappa_{lk}. \tag{1.1}$$

We can contract this identity with $(\kappa^{-1})_{im}(\kappa^{-1})_{kn}$ to get

$$(\kappa^{-1})_{kn}c_{jkm}=(\kappa^{-1})_{im}c_{ijn}\Rightarrow(\kappa^{-1})_{in}c_{jim}+(\kappa^{-1})_{im}c_{jin}=0 \tag{1.2}$$

Now let $C=-(\kappa^{-1})_{ij}T_iT_j$. We have

$$[C,T_k]=(\kappa^{-1})_{ij}[T_k,T_iT_j]=(\kappa^{-1})_{ij}([T_k,T_i]T_j+T_i[T_k,T_j])=(\kappa^{-1})_{ij}(c_{kil}T_lT_j+c_{kjl}T_iT_l). \tag{1.3}$$

Relabelling indices,

$$[T_k,C]=((\kappa^{-1})_{ij}c_{kil}+(\kappa^{-1})_{il}c_{kij})T_lT_j=0 \tag{1.4}$$

using (1.2).

Question 2

We have $g=a_0I+ia_i\sigma_i$, so $g^{-1}=g^\dagger=a_0I-ia_i\sigma_i$ and $dg=da_0I+ida_i\sigma_i$. The constraint $a_0^2+a_ia_i=1$ implies that $a_0da_0+a_ida_i=0$.

Using $\sigma_i\sigma_j=\delta_{ij}I+i\varepsilon_{ijk}\sigma_k$ we can compute

$$\begin{aligned}dgg^{-1}&=(da_oI+ida_i\sigma_i)(a_0I-ia_j\sigma_j)\\&=a_0da_0+a_ida_i+i(a_0da_k-a_kda_0+\varepsilon_{ijk}da_ia_j)\sigma_k\\&=i(a_0da_k-a_kda_0+\varepsilon_{ijk}da_ia_j)\sigma_k\end{aligned} \tag{2.1}$$

using the constraint in the last step. This is obviously of the right form to be in the Lie algebra.

Using $\mathrm{tr}(\sigma_k\sigma_l)=2\delta_{kl}$ we have

$$\begin{aligned}-\frac{1}{2}\mathrm{tr}(dgg^{-1}dgg^{-1})&=(a_0da_k-a_kda_0+\varepsilon_{ijk}da_ia_j)(a_0da_k-a_kda_0+\varepsilon_{lmk}da_ka_m)\\&=a_0^2da_kda_k-2a_0a_kda_0da_k+\underbrace{a_ka_k}_{=1-a_0^2}da_0da_0+\underbrace{\varepsilon_{ijk}\varepsilon_{lmk}}_{=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}}da_ia_jda_la_m\\&=\underbrace{((a_0)^2+a_ia_i)}_{=1}da_kda_k+da_0^2\underbrace{-a_ida_ia_jda_j-2a_0da_0a_kda_k-a_0da_0a_0da_0}_{=-(a_0da_0+a_ida_i)^2=0}\\&=da_0^2+da_ida_i\end{aligned} \tag{2.2}$$

as expected.

Question 3

Let us recall we have the following generators for SU(3):

$$\begin{aligned}&e_\alpha=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\quad e_{-\alpha}=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\quad e_\beta=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\quad e_{-\beta}=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\\&e_\gamma=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\quad e_{-\gamma}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\quad h_1=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad h_2=\frac{1}{2\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\end{aligned} \tag{3.1}$$

We let $\vec{h}=(h_1,h_2)$, then $[\vec{h},e_{\pm\alpha}]=\pm\vec{\alpha}e_{\pm\alpha}$ and so on, with

$$\vec{\alpha}=(1,0) \quad \vec{\beta}=\left(-\frac{1}{2},+\frac{\sqrt{3}}{2}\right) \quad \vec{\gamma}=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right). \tag{3.2}$$

Note that $\vec{\gamma}=\vec{\alpha}+\vec{\beta}$.

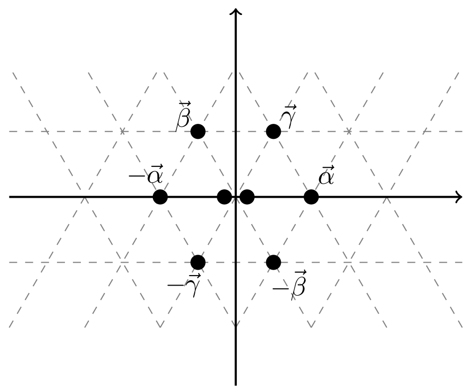

Figure 1: The SU(3) root diagram.

We now consider the Jacobi identity in the form

$$[\vec{h},[e_\alpha,e_\beta]]+[e_\alpha,[e_\beta,\vec{h}]]+[e_\beta,[\vec{h},e_\alpha]]=0 \tag{3.3}$$

which implies

$$[\vec{h},[e_\alpha,e_\beta]]=\left(\vec{\alpha}+\vec{\beta}\right)[e_\alpha,e_\beta]=\vec{\gamma}[e_\alpha,e_\beta] \tag{3.4}$$

which must mean that $[e_\alpha,e_\beta]$ is proportional to the eigenmatrix of $\vec{h}$ with eigenvalue $\vec{\gamma}$, i.e. to e_γ. Indeed direct computation using the basis given above shows that $[e_\alpha,e_\beta]=e_\gamma$.

Similarly, one can evaluate

$$[\vec{h},[e_\alpha,e_\gamma]]=(\vec{\alpha}+\vec{\gamma})[e_\alpha,e_\gamma]. \tag{3.5}$$

Now there is no vector $\vec{\alpha}+\vec{\gamma}$ in the root diagram 1, i.e. there is no eigenmatrix of $\vec{h}$ with this eigenvalue, so this equation only makes sense if in fact $[e_\alpha,e_\gamma]=0$. This is again confirmed by direct computation.

Finally, one has

$$[\vec{h},[e_{-\beta},e_\beta]]=0, \tag{3.6}$$

which implies that $[e_{-\beta},e_\beta]$ is either zero, or else is actually in the Cartan subalgebra, i.e. it can be expressed as a linear combination of h_1 and h_2. Indeed, explicitly we find

$$[e_{-\beta},e_\beta]=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&1\end{pmatrix}\overset{!}{=}c_1h_1+c_2h_2=\frac{1}{2}\begin{pmatrix}c_1+\frac{1}{\sqrt{3}}c_2&0&0\\0&-c_1+\frac{1}{\sqrt{3}}c_2&0\\0&0&-\frac{2}{\sqrt{3}}c_2\end{pmatrix}. \tag{3.7}$$

We must have $c_2=-\sqrt{3}$ and $c_1=1$, so in fact $[e_{-\beta},e_\beta]=-2\vec{\beta}\cdot\vec{h}$.

Question 4

For this question and from now on we're going to choose to work with a $1/\sqrt{2}$ rescaling of the generators that are not in the Cartan subalgebra, i.e. $e_{\pm\alpha}\to 1/\sqrt{2}e_{\pm\alpha}$. With this rescaling we get the the three $L(\mathrm{SU}(2))$ subalgebras

$$\begin{aligned}&[h_\alpha,e_{\pm\alpha}]=\pm e_{\pm\alpha} \quad [e_\alpha,e_{-\alpha}]=h_\alpha\\&[h_\beta,e_{\pm\beta}]=\pm e_{\pm\beta} \quad [e_\beta,e_{-\beta}]=h_\beta\\&[h_\gamma,e_{\pm\gamma}]=\pm e_{\pm\gamma} \quad [e_\gamma,e_{-\gamma}]=h_\gamma\end{aligned} \tag{4.1}$$

where $h_\alpha=\vec{\alpha}\cdot\vec{h}$. Each of these subalgebras is invariant under reflection of their associated roots, i.e. under $\vec{\alpha}\to-\vec{\alpha}$. In any representation of $L(\mathrm{SU}(2))$ this reflection invariance leads to the reflection symmetry of the weights, $m\to-m$ as we have to send $h_\alpha\to-h_\alpha$. As the weights of $L(\mathrm{SU}(3))$ are the eigenvalues of h_1 and h_2 this automatically gives reflection symmetries of all $L(\mathrm{SU}(3))$ weight diagrams.

It is easy to see this geometrically. Any finite dimensional representation of $L(\mathrm{SU}(3))$ can be generated by starting with a highest weight state and acting with the lowering generators $e_{-\alpha}$, $e_{-\beta}$, $e_{-\gamma}$ on the weight. Geometrically the combined action of the raising and lowering operators is shown in figure 2. The reflection symmetry in each root then means that the diagram will be reflection symmetric under each of $\vec{\alpha}\to-\vec{\alpha}$, $\vec{\beta}\to-\vec{\beta}$, $\vec{\gamma}\to-\vec{\gamma}$.

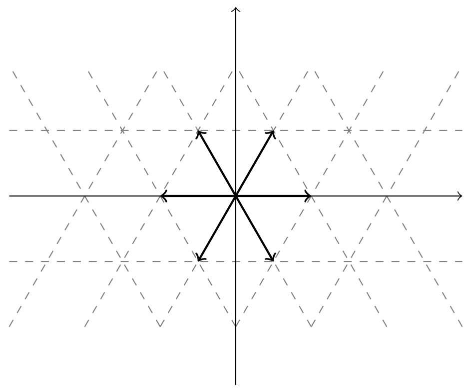

Figure 2: The SU(3) raising and lowering operators act in the indicated directions, giving reflection symmetry in the lines through the origin orthogonal to each of the three raising/lowering directions shown.

To work out these reflection symmetries, note that the image of a point $\vec{p}$ under reflection through a unit vector $\vec{m}$ is $\vec{p}'=-\vec{p}+2\vec{m}\vec{p}\cdot\vec{m}$. To see this, consider the setup shown in figure 3. The intersection of the dashed line and $\vec{m}$ gives the vector $\hat{m}\vec{p}\cdot\hat{m}$, where $\hat{m}=\vec{m}/|\vec{m}|$. Then by vector addition the point $\vec{p}'$ is easily seen to be $\hat{m}\vec{p}\cdot\hat{m}-(\vec{p}-\hat{m}\vec{p}\cdot\hat{m})=-\vec{p}+2\hat{m}\vec{p}\cdot\hat{m}$.

Therefore for each of the roots we have the following.

We specify the line through the origin orthogonal to $\vec{\alpha}=(1,0)$ by the vector $\vec{m}=(0,1)$, reflection in $\vec{m}$ then means reflection in the h_2 (vertical) axis, and so if (p,q) is a weight so too is $(-p,q)$.

For $\vec{\beta}=(-1/2,\sqrt{3}/2)$ we reflect in $\vec{m}=(\sqrt{3}/2,1/2))$, i.e. through the line through the origin at an angle $\theta=\pi/6$. Using the above formula, if (p,q) is a weight so too is $((p+\sqrt{3})q/2,(\sqrt{3}p-q)/2)$.

For $\vec{\gamma}=(1/2,\sqrt{3}/2)$ we reflect in $\vec{m}=(-\sqrt{3}/2,1/2))$, i.e. through the line through the origin at an angle $\theta=5\pi/6$. Using the above formula, if (p,q) is a weight so too is $((p-\sqrt{3})q/2,(-\sqrt{3}p-q)/2)$.

Using the above, or by looking at the root diagram, one finds that reflection in any of $\vec{\alpha},\vec{\beta}$ or $\vec{\gamma}$ has the effect of interchanging the other two roots (up to minus signs). In particular under the reflection sending $\vec{\alpha}\to-\vec{\alpha}$ we have $\vec{\beta}\to\vec{\gamma},\vec{\gamma}\to\vec{\beta}$, under the reflection sending $\vec{\beta}\to-\vec{\beta}$ we have $\vec{\alpha}\to\vec{\gamma},\vec{\gamma}\to\vec{\alpha}$, and the under the reflection sending $\vec{\gamma}\to-\vec{\gamma}$ we have $\vec{\alpha}\to-\vec{\beta},\vec{\beta}\to-\vec{\alpha}$. As the Lie algebra and its commutation relations is completely specified by its roots this amounts to a symmetry of the algebra. This symmetry will then be inherited by all its representations.

If we restrict to a particular $L(\mathrm{SU}(2))$ subalgebra then an irreducible representation of $L(\mathrm{SU}(3))$ decomposes into a set of irreducible representations of $L(\mathrm{SU}(2))$. If $\vec{\lambda}$ is a weight of $L(\mathrm{SU}(3))$ then $\vec{\alpha}\cdot\vec{\lambda}$ will be the corresponding weight of the $\vec{\alpha}$ $L(\mathrm{SU}(2))$ subalgebra, and so on. Because we can use (compositions of) the reflection symmetries above to send any particular (non-zero) root to any other we can in fact map each of the $\vec{\alpha},\vec{\beta},\vec{\gamma}$ $L(\mathrm{SU}(2))$ subalgebras into each other, establishing that the sets of $L(\mathrm{SU}(2))$ irreps obtained from any of them are in fact the same.

As an example, consider the fundamental representation, which is three-dimensional. The weights (from reading off the diagonal

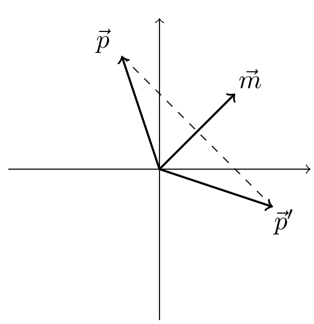

Figure 3: Reflection of the point $\vec{p}$ through the vector $\vec{m}$.

entries of the Cartan subalgebra) are $\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(0,-\frac{1}{\sqrt{3}}\right)$ and the weight diagram of figure 4 is easily seen to have the expected reflection symmetry. Restricting to any of the $L(\mathrm{SU}(2))$ subalgebras we obtain a decomposition $\mathbf{3}\to\mathbf{2}\oplus\mathbf{1}$ as we obtain a pair of reflection symmetric weights, corresponding to the two-dimensional irrep of $L(\mathrm{SU}(2))$, as well as an additional weight corresponding to the trivial representation. This can be seen both by looking at the weight diagram and by showing that the $L(\mathrm{SU}(2))$ weights $\vec{\alpha}\cdot\vec{\lambda}$, $\vec{\beta}\cdot\vec{\lambda}$, $\vec{\gamma}\cdot\vec{\lambda}$ are all $\pm1/2,0$.

Similarly, consider the six-dimensional irrep of figure 5. For each $L(\mathrm{SU}(2))$ subalgebra we have the decomposition $\mathbf{6}\to\mathbf{3}\oplus\mathbf{2}\oplus1$, as is easy to see geometrically.

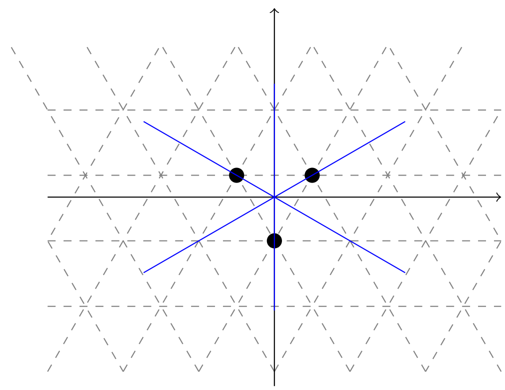

Figure 4: The weights of the SU(3) fundamental representation. Axes of reflection symmetry in blue.

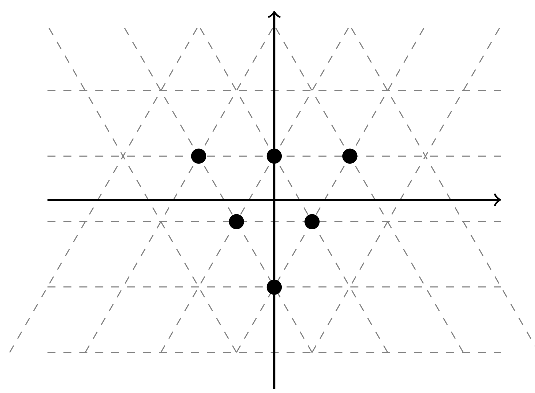

Figure 5: The weights of the **6** of SU(3).

Question 5

The states of the **3** can be called u, d and s with weights $\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(0,-\frac{1}{\sqrt{3}}\right)$ respectively (these are just the eigenvalues of the Cartan matrices h_1 and h_2 in the fundamental representation). The triple tensor product $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ consists of all possible triple tensor products of these states, and the weights in the product are of course given by the sums of the individual weights. We can write down all the possible states and weights fairly easily. Note we'll use the shorthand $uds\equiv u\otimes d\otimes s$ i.e. the tensor product will be implicit. Then we have the states and weights in table 1.

Now, we need to know the action of the lowering operators on the states u, d and s. By looking at the weights we find that $e_{-\alpha}$ lowers the weight by $\vec{\alpha}=(1,0)$ and sends $u\to d$, $e_{-\beta}$ lowers the weight by subtracting $\vec{\beta}=(-1/2,+3/2\sqrt{3})$ and sends $d\to s$ and also $e_{-\gamma}$ lowers the weight by subtracting $\vec{\gamma}=(1/2,3/2\sqrt{3})$ and sends $u\to s$. As $\vec{\gamma}=\vec{\alpha}+\vec{\beta}$ we need only use the former two. In the tensor product the lowering operators are

$$E_{-\alpha}=e_{-\alpha}\otimes I\otimes I+I\otimes e_{-\alpha}\otimes I+I\otimes I\otimes e_{-\alpha} \tag{5.1}$$

and similarly for $E_{-\beta}$.

Seeing as u is the highest weight state of the **3** representation it is annihilated by all the raising operators and hence uuu is a highest weight state of the tensor product representation. We can build an irreducible representation from it by acting with

uuu	$(3/2,3/2\sqrt{3})$
sss	$(0,-3/\sqrt{3})$
ddd	$(-3/2,3/2\sqrt{3})$
uud,udu,duu	$(1/2,3/2\sqrt{3})$
uus,usu,suu	$(1,0)$
uds,usd,dsu,dus,sdu,sud	$(0,0)$
ddu,dud,udd	$(-1/2,3/2\sqrt{3})$
ssu,sus,uss	$(1/2,-3/2\sqrt{3})$
ssd,sds,dss	$(-1/2,-3/2\sqrt{3})$
dds,dsd,sdd	$(-1,0)$

Table 1: The states and weights of the **3**⊗**3**⊗**3** tensor product.

all possible combinations of $E_{-\alpha}$ and $E_{-\beta}$. This produces the following 10-dimensional representation:

Act with:	uuu	$(3/2,3/2\sqrt{3})$
$E_{-\alpha}$	$uud+udu+duu$	$(1/2,3/2\sqrt{3})$
$E_{-\alpha}E_{-\alpha}$	$ddu+dud+udd$	$(-1/2,3/2\sqrt{3})$
$E_{-\alpha}E_{-\alpha}E_{-\alpha}$	ddd	$(-3/2,3/2\sqrt{3})$
$E_{-\beta}E_{-\alpha}$	$uus+usu+suu$	$(1,0)$
$E_{-\beta}E_{-\alpha}E_{-\alpha}$	$uds+usd+dsu+dus+sdu+sud$	$(0,0)$
$E_{-\beta}E_{-\beta}E_{-\alpha}E_{-\alpha}$	$ssu+sus+uss$	$(1/2,-3/2\sqrt{3})$
$E_{-\beta}E_{-\alpha}E_{-\alpha}E_{-\alpha}$	$dds+dsd+sdd$	$(-1,0)$
$E_{-\beta}E_{-\beta}E_{-\alpha}E_{-\alpha}E_{-\alpha}$	$ssd+sds+dss$	$(-1/2,-3/2\sqrt{3})$
$E_{-\beta}E_{-\beta}E_{-\beta}E_{-\alpha}E_{-\alpha}E_{-\alpha}$	sss	$(0,-3/\sqrt{3})$

Table 2: The states and weights of the **10** in the **3**⊗**3**⊗**3** tensor product.

This removes 10 states from the initial 27 of table 1. We are left with the states in table 3, where it is understood that at each weight we need to take appropriate linear combination of the states which must be orthogonal to the state of the same weight in the **10**.

uud,udu,duu	$(1/2,3/2\sqrt{3})$
uus,usu,suu	$(1,0)$
uds,usd,dsu,dus,sdu,sud	$(0,0)$
ddu,dud,udd	$(-1/2,3/2\sqrt{3})$
ssu,sus,uss	$(1/2,-3/2\sqrt{3})$
ssd,sds,dss	$(-1/2,-3/2\sqrt{3})$
dds,dsd,sdd	$(-1,0)$

Table 3: The states and weights of the **3**⊗**3**⊗**3** tensor product after removing the **10**.

The highest weight available appears to be now $(1/2,3/2\sqrt{3})$ (as can be confirmed by noting where this would appear on the weight diagram, there is no other weight left which could be reached by adding one of the positive roots). Seeing as there is one state of this weight in the **10** we expect there should be two independent linear combinations of uud, udu, duu left, which should be annihilated by $E_{+\alpha}$, $E_{+\beta}$, $E_{+\gamma}$. The only non-zero relation involving a raising operator and d is $e_\alpha d=u$. So writing

$$0=E_\alpha(c_1uud+c_2udu+c_3duu)=(c_1+c_2+c_3)uuu\Rightarrow c_1=-(c_2+c_3) \tag{5.2}$$

which is also the condition for such a state to be orthogonal to the state $uud+udu+duu$ appearing in the **10**. There are obviously many solutions to this. Let's take the two orthogonal states $uud-udu$ and $uud+udu-2duu$. Each of these gives the highest weight state for an irreducible representation. The former gives table 4 and the latter gives table 5.

The **10** and the two **8**s use up all except one state, which is the left-over linearly independent state of weight (0,0) which must correspond to the trivial representation **1**. The precise form of this state is found by looking for the unique linear combination of $uds, usd, dsu, dus, sdu, sud$ which is annihilated by all the raising and lowering operators. Applying E_α to $c_1uds+c_2usd+c_3dsu+c_4dus+c_5sud+c_6sdu$ we find we want $c_1=-c_4$, $c_2=-c_3$, $c_5=-c_6$, and applying E_β we get the conditions $c_1=-c_2$, $c_6=-c_3$ and $c_4=-c_5$ which fixes the state in the singlet to be (up to normalisation) $uds-usd+dsu-dus+sud-sdu$,

Act with:	$uud-udu$	$(1/2,3/2\sqrt{3})$
$E_{-\alpha}$	$dud-ddu$	$(-1/2,3/2\sqrt{3})$
$E_{-\beta}E_{-\alpha}$	$sud+dus-sdu-dsu$	$(0,0)$
$E_{-\alpha}E_{-\beta}E_{-\alpha}$	$dds-dsd$	$(-1,0)$
$E_{-\beta}E_{-\beta}E_{-\alpha}$	$sus-ssu$	$(1/2,-3/2\sqrt{3})$
$E_{-\alpha}E_{-\beta}E_{-\beta}E_{-\alpha}$	$sds-ssd$	$(-1/2,-3/2\sqrt{3})$
$E_{-\beta}$	$uus-usu$	$(1,0)$
$E_{-\alpha}E_{-\beta}$	$dus+uds-sdu-sud$	$(0,0)$

Table 4: The states and weights of an **8** in the **3**$\otimes$**3**$\otimes$**3** tensor product. Note that acting with $E_{-\beta}E_{-\alpha}E_{-\beta}E_{-\alpha}$ would give the same as acting as acting with $E_{-\alpha}E_{-\beta}E_{-\beta}E_{-\alpha}$.

Act with:	$uud+udu-2duu$	$(1/2,3/2\sqrt{3})$
$E_{-\alpha}$	$2udd-dud-ddu$	$(-1/2,3/2\sqrt{3})$
$E_{-\beta}E_{-\alpha}$	$2uds+2usd-sud-dus-sdu-dsu$	$(0,0)$
$E_{-\alpha}E_{-\beta}E_{-\alpha}$	$dds+dsd-2sdd$	$(-1,0)$
$E_{-\beta}E_{-\beta}E_{-\alpha}$	$uss-sus-ssu$	$(1/2,-3/2\sqrt{3})$
$E_{-\alpha}E_{-\beta}E_{-\beta}E_{-\alpha}$	$2dss-sds-ssd$	$(-1/2,-3/2\sqrt{3})$
$E_{-\beta}$	$uus+usu-2suu$	$(1,0)$
$E_{-\alpha}E_{-\beta}$	$dus+usd+dsu+usd-2sdu-2sud$	$(0,0)$

Table 5: The states and weights of the second **8** in the **3**$\otimes$**3**$\otimes$**3** tensor product. Note that acting with $E_{-\beta}E_{-\alpha}E_{-\beta}E_{-\alpha}$ would give the same as acting as acting with $E_{-\alpha}E_{-\beta}E_{-\beta}E_{-\alpha}$.

which is totally antisymmetric in u,d,s.

Hence we have completed showing the decomposition $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$.

In the quark model we identify points (p,q) with $(I_3,\sqrt{3}/2Y)$ where I_3 is the isospin and Y is the hypercharge. Thus the baryon singlet state, which has quark content uds, has as quantum numbers $I_3=0$ and $Y=0$. It sits in a totally antisymmetric flavour representation and by colour confinement also in a totally antisymmetric colour representation. The product of these two wave functions is then symmetric, however Fermi statistics imply the overall wave function must be antisymmetric (quarks are spin 1/2). Now if we take the tensor product of three spin 1/2 states, which as $1/2\otimes1/2=0\oplus1$ consists of two spin 1/2 representations and one spin 3/2 representation, the totally symmetric spin up state constituents the highest weight state of the spin 3/2 representation, and so this representation must be symmetric. Therefore the baryon singlet can't have spin 3/2 and should instead have spin 1/2. However the spin 1/2 representations in the decomposition are in fact of mixed symmetry rather than being antisymmetric. Thus we cannot combine them with the antisymmetric flavour and colour wavefunctions to obtain a totally antisymmetric wavefunction. Thus we conclude that the baryon singlet does not exist in the ground state. (It can exist in excited states with non-zero orbital angular momentum, though.)

This state differs from the Λ_0 state (which is one of the (0,0) states in the octet) in that it is a totally antisymmetric combination of uds whereas the Λ_0 state is of mixed symmetry. That the Λ_0 state has mixed symmetry allows it to exist in the ground state (in a spin 1/2 representation).

Question 6

a) Let I be an ideal of L a compact finite-dimensional Lie algebra, and $I_\perp$ the orthogonal complement with respect to the Killing form. Let $X\in I$, $Y\in L$ and $Z\in I_\perp$. Then

$$\kappa(X,[Y,Z])=\mathrm{tr}(XYZ-XZY)=\mathrm{tr}(XYZ-YXZ)=\kappa([X,Y],Z). \tag{6.1}$$

Now as I is an ideal $[X,Y]\in I$, and as $Z\in I_\perp$ the above vanishes. We see then that $[Y,Z]$ is orthogonal to all $X\in I$ and so must lie in $I_\perp$, which tells us that $I_\perp$ is also an ideal.

The definition of I and $I_\perp$ as orthogonal complements gives the vector space decomposition $L=I\oplus I_\perp$. To see that these summands mutually commute take $X\in I$ and $Z\in I_\perp$. Then we consider for arbitrary $Y\in L$

$$\kappa([X,Z],Y)=\kappa(Z,[Y,X])=0, \tag{6.2}$$

where we use at the end that $[Y,X]\in I$. Then the non-degeneracy of the Killing form (Lie algebra of compact type implies it is negative definite) this implies that actually $[X,Z]=0$ identically. Finally we observe that as both I and $I_\perp$ are closed under commutation with themselves they are in fact also Lie algebras (of compact type).

By iterating these results until within both I, $I_\perp$ we can find no further ideals, we see that L can be expressed as a direct sum of simple Lie algebras of compact type.

b) If L is simple then it is non-abelian and has no proper ideals, i.e. no ideals which are not L itself or the trivial (zero) ideal $\emptyset$. As it is non-abelian at least one of the commutators of its elements must be non-zero, so $[L,L]$ is not $\emptyset$. It must be then that $[L,L]=L$ itself, as if we had $[L,L]=I\subset L$ then it trivially follows that $[L,I]\subset I$, and so it would have a proper ideal, contrary to assumption.

We know from question 3 a basis for the Lie algebra of SU(3). It is straightforward to check that the commutation relations of this basis generate again all the elements of the basis and no new ones, so that $[L,L]=L$ in this case.

For U(3) the Lie algebra contains an additional generator proportional to the identity (this is because we do not impose the tracelessness condition on the Lie algebra, this new generator represents the trace part of an antihermitian matrix). This generator commutes with all the $L(\mathrm{SU}(3))$ generators, and does not appear on the right-hand side of any commutators itself. As a result we have instead $[L(\mathrm{U}(3)),L(\mathrm{U}(3))]=L(\mathrm{SU}(3))$.

Question 7

Suppose that d is an irrep of $L(\mathrm{SU}(3))$ acting on V and let $v\in V$ have weight $\vec{\lambda}$, i.e. $d(\vec{h})v=\vec{\lambda}v$. Then assuming $d(e_\alpha)v$ is non-zero

$$d(\vec{h})d(e_\alpha)v=[d(\vec{h}),d(e_\alpha)]v+d(e_\alpha)d(\vec{h})v=\left(\vec{\lambda}+\vec{\alpha}\right)d(e_\alpha)v, \tag{7.1}$$

using $[d(\vec{h}),d(e_\alpha)]=d([\vec{h},e_\alpha])=\vec{\alpha}d(e_\alpha)$. Obviously entirely similar results hold for the other weights. By starting with v we are able to generate the whole irrep by acting with $d(e_{\pm\alpha}),d(e_{\pm\beta}),d(e_{\pm\gamma})$ and so the weights of all the elements of V then differ by linear combinations of the roots with integer coefficients, i.e. by elements of the root lattice.

The weight lattice is the lattice of all possible weights. As roots are the weights of the adjoint representation the root lattice is a subset of the weight lattice. We know that the fundamental representation of $L(\mathrm{SU}(3))$ has weights $\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(0,-\frac{1}{\sqrt{3}}\right)$. Shifts by root vectors generates the lattice shown in figure 7. Similarly the antifundamental representation has weights $\left(-\frac{1}{2},-\frac{1}{2\sqrt{3}}\right),\left(\frac{1}{2},-\frac{1}{2\sqrt{3}}\right),\left(0,\frac{1}{\sqrt{3}}\right)$, and shifts by root vectors generates the lattice shown in figure 8.

Higher-dimensional irreducible representations can be generated by taking tensor products of the fundamental and the antifundamental. To see this we note that the highest weights of these representations are $\vec{\lambda}_1\equiv\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)$ and $\vec{\lambda}_2\equiv\left(0,\frac{1}{\sqrt{3}}\right)$. If we let $\vec{\alpha}_1=\vec{\alpha}$ and $\vec{\alpha}_2=\vec{\beta}$ denote the simple roots of $L(\mathrm{SU}(3))$ then the weights $\vec{\lambda}_i$ obey $2\vec{\alpha}_i\cdot\vec{\lambda}_j=\delta_{ij}$. Now, any irreducible representation has a highest weight $\vec{\lambda}$ obeying $2\vec{\alpha}_i\cdot\vec{\lambda}\in\mathbb{Z}$. It follows we can decompose $\vec{\lambda}=n_i\vec{\lambda}_i$, where the coefficients n_i are given by $n_i=2\vec{\alpha}_i\cdot\vec{\lambda}$ and so are integers. Hence any highest weight can be written as a linear combination with integer coefficients of the weights of the fundamental and antifundamental. So by taking a tensor product of n_1 copies of the $\mathbf{3}$ and n_2 copies of the $\overline{\mathbf{3}}$ we can generate a reducible representation including any given highest weight $\vec{\lambda}$. Thus all higher-dimensional irreducible representations can be generated by these tensor products.

Now, a generic element of the weight lattice associated to the fundamental is of the form

$$\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)+N_1\vec{\alpha}+N_2\vec{\beta}. \tag{7.2}$$

If we take the tensor product of two representations whose weights lie in this lattice then we will obtain weights of the form

$$\left(1,\frac{1}{\sqrt{3}}\right)+(N_1+N_1')\vec{\alpha}+(N_2+N_2')\vec{\beta}=\left(0,\frac{1}{\sqrt{3}}\right)+(N_1+N_1'+1)\vec{\alpha}+(N_2+N_2')\vec{\beta} \tag{7.3}$$

which lie in the weight lattice associated to the antifundamental representation. Similarly we can show that the tensor product of two representations whose weights lie in the lattice associated to the antifundamental gives weights lying in that associated to the fundamental, and that the tensor product of a fundamental and an antifundamental gives us weights lying in the root lattice. In this way we see that all possible weights lie either in the root lattice or one of the two lattices of figures 7 and 8. It's easy to confirm that the latter are just translations of the root lattice by elements from the weight space of the $\mathbf{3}$ and $\overline{\mathbf{3}}$. In the first case we translate by $(0,-\frac{1}{\sqrt{3}})$, and in the second by $(0,\frac{1}{\sqrt{3}})$.

To find the centre of SU(3) we should use Schur's lemma from the start, noting that SU(3) itself defines the fundamental representation. So by Schur's lemma anything that commutes with everything in the an irreducible representation must be of the form λI: in the fundamental representation I is the three-by-three identity and to have a special unitary matrix we need $|\lambda|^2=1$ and $\det(\lambda I)=\lambda^3=1$. The only elements of the centre are therefore $I,\omega I$ and $\omega^2 I$ where $\omega=e^{2\pi i/3}$ is a cubic root of unity. Schur's lemma and the homomorphism property of group representations ($D(g)D(h)=D(gh)=D(hg)=D(h)D(g)$ for g an arbitrary group element and h in the centre) of course also imply that the elements of the centre act as multiplication by a constant in any representation.

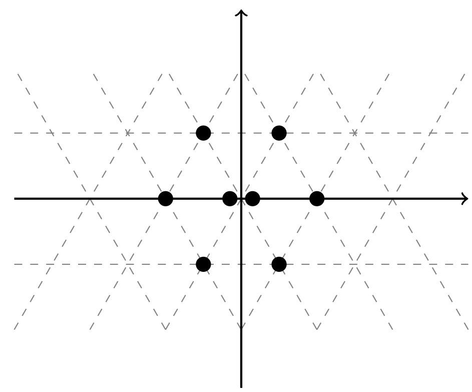

Figure 6: The SU(3) root lattice.

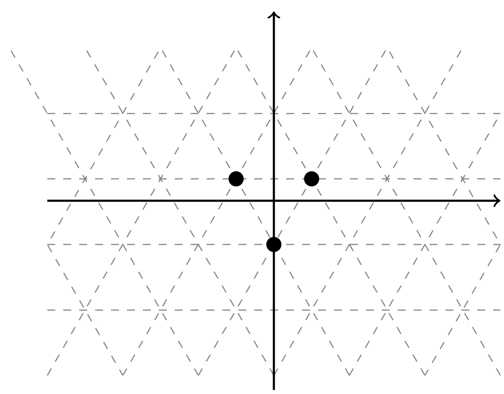

Figure 7: The weights of SU(3) fundamental representation. The lattice corresponds to shifts by root vectors.

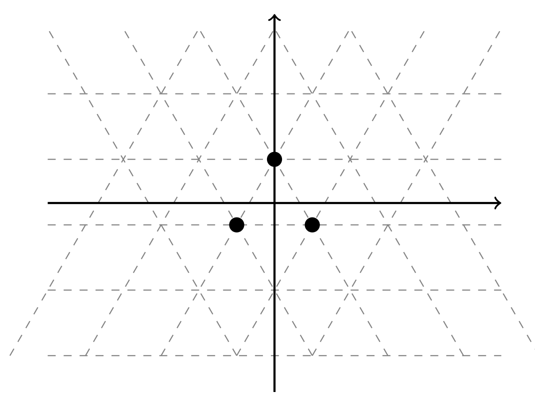

Figure 8: The weights of SU(3) anti-fundamental representation. The lattice corresponds to shifts by root vectors.

We can characterise the weights in terms of the action of the centre as follows. Let $Z \equiv \omega I$. Acting on the **3** representation Z acts as a scaling by ω. Acting on the complex conjugate of this representation, the $\overline{\mathbf{3}}$, Z clearly acts as a scaling by $\omega^* = \omega^2$. Acting on the **8** Z acts as the identity - this is because the **8** is the adjoint and an element g of the group acts on the adjoint by $X \to gXg^{-1}$. We therefore characterise the weights by the power of ω in the above action of the centre. This power can be 1, -1 or 0. This characterisation is known as triality. (To complete this argument you should convince yourself that the Z acts in the same way for any representation which has weights in the **3** lattice as it does on the **3**, etc, using the fact that we can generate all highest weights by appropriate tensor products of the fundamental and its conjugate, on which we know the action of Z.)

Question 8

The Lie algebra of SO(3) consists of real antisymmetric matrices. A standard basis is

$$T_1=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\quad T_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\quad T_3=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}. \tag{8.1}$$

One can check that T_3 satisfies $T_3^3=-T_3$ so $(iT_3)^3=(iT_3)$ and therefore iT_3 has eigenvalues $\pm1,0$. Therefore by suitable conjugation it can be taken to the diagonal form

$$iT_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}. \tag{8.2}$$

From question 3 it is easy to see that this equals $2h_\alpha$ where $h_\alpha\equiv\vec{\alpha}\cdot\vec{h}=h_1$. As the eigenvalues of h_α in any representation are integral or half-integral (because it is part of an $L(\mathrm{SU}(2))$ subalgebra of $L(\mathrm{SU}(3))$) the eigenvalues of iT_3 are all integral (if $d(h_\alpha)v=\lambda v$ then $id(T_3)v=2d(h_\alpha)v=2\lambda v$).

In order to find the decomposition of $L(\mathrm{SU}(3))$ irreps into $L(\mathrm{SO}(3))$ irreps we just need to look at the h_1 values of all weights in the irreps - multiplying these by two will give us the weights of $L(\mathrm{SO}(3))$ representations.

For instance, the **3** representation has $L(\mathrm{SU}(3))$ weights

$$\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\left(0,-\frac{1}{\sqrt{3}}\right). \tag{8.3}$$

This gives $L(\mathrm{SO}(3))$ weights $1,-1,0$, which we identify as the three-dimensional fundamental representation of $L(\mathrm{SO}(3))$.

The $\overline{\mathbf{3}}$ representation has $L(\mathrm{SU}(3))$ weights

$$\left(-\frac{1}{2},-\frac{1}{2\sqrt{3}}\right),\left(\frac{1}{2},-\frac{1}{2\sqrt{3}}\right),\left(0,\frac{1}{\sqrt{3}}\right). \tag{8.4}$$

The $L(\mathrm{SO}(3))$ weights are again $1,-1,0$, giving the three-dimensional fundamental representation of $L(\mathrm{SO}(3))$. It makes sense that both the complex conjugates **3** and $\overline{\mathbf{3}}$ restrict to the same representation of $L(\mathrm{SO}(3))$, because the latter is real.

The **6** representations has $L(\mathrm{SU}(3))$ weights

$$\left(1,\frac{1}{\sqrt{3}}\right),\left(1/2,-\frac{1}{2\sqrt{3}}\right),\left(0,\frac{1}{\sqrt{3}}\right),\left(0,-\frac{1}{2\sqrt{3}}\right),\left(-1/2,-\frac{1}{2\sqrt{3}}\right),\left(-1,\frac{1}{\sqrt{3}}\right). \tag{8.5}$$

This gives $L(\mathrm{SO}(3))$ weights $2,1,0,0,-1,2$. From these we pick out the five-dimensional representation with weights $2,1,0,-1,-2$ and the trivial representation 0, i.e. we have the restriction $\mathbf{6}\to\mathbf{5}\oplus\mathbf{1}$. The $\overline{\mathbf{6}}$ representation will restrict in the same way.

The **8** representation (which is the adjoint and is real) has $L(\mathrm{SU}(3))$ weights

$$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),(1,0),\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),(0,0),(0,0),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right),(-1,0),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right), \tag{8.6}$$

The $L(\mathrm{SO}(3))$ weights are $1,2,-1,0,0,1,-2,1$. This time we can pick out the five-dimensional representation with weights $2,1,0,-1,-2$ as well as the three-dimensional representation $1,0,-1$. So $\mathbf{8}\to\mathbf{5}\oplus\mathbf{3}$.

Finally the **10** representation has $L(\mathrm{SU}(3))$ weights

$$\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),(1,0),\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\left(0,-\sqrt{3}\right),(0,0),\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),(-1,0),\left(-\frac{3}{2},\frac{\sqrt{3}}{2}\right). \tag{8.7}$$

We get $L(\mathrm{SO}(3))$ weights $3,2,1,1,0,0,-1,-1,-2,-3$. This gives the seven-dimensional representation with weights $3,2,1,0,-1,-2,-3$ and the three-dimensional representation $1,0,-1$. So $\mathbf{10}\to\mathbf{7}\oplus\mathbf{3}$.

You can check that this is all consistent with SO(3) tensor products. For instance, we know that the tensor product of two SO(3) vectors can be decomposed into a symmetric traceless part, an antisymmetric part, and a trace. The antisymmetric representation has dimension $3\times2/2=3$ and the symmetric traceless part has dimensions $3\times4/2-1=5$. On the SU(3) side we know that $\mathbf{3}\otimes\mathbf{3}=\mathbf{6}\oplus\overline{\mathbf{3}}$. This indeed restricts to the SO(3) expression $\mathbf{3}\otimes\mathbf{3}=\mathbf{5}\oplus\mathbf{1}\oplus\mathbf{3}$. Similarly $\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{8}\oplus\mathbf{1}$ for SU(3) gives the same result, as it must. You can probably check more such products if you really want.

Question 9

Suppose G/H is a homogeneous space. It will be a symmetric space if there is a Lie algebra decomposition $L(G)=L(H)\oplus M$ such that

$$[L(H),L(H)]\subset L(H)\quad [L(H),M]\subset M\quad [M,M]\subset L(H). \tag{9.1}$$

The second condition is generally true for a homogeneous space as it is the tangent space statement of the fact that the group H is the stabilser group of some point in the homogeneous space. If we decompose the Lie algebra $L(G)$ (which is the tangent space of G) into $L(H)$ and M then roughly speaking M should correspond to the tangent space of the homogeneous space itself, which one would generally expect to be invariant under the action of the Lie algebra of H. The map $M \to -M$ (i.e. flip the sign of all elements of M) is obviously a symmetry compatible with these brackets. The geometric interpretation is as a sort of reflection symmetry about a point, i.e. given a point $x \in G/H$ it preserves x and acts as minus the identity on the tangent space at x. (This has an alias as a geodesic reflection symmetry, where a geodesic $\gamma(t) \to \gamma(-t)$). This makes sense from the Lie algebra decomposition, viewing M as the tangent space to G/H at the origin in the same way that $L(G)$ would be the tangent space to G at the origin.
a) Consider the Lie algebra of SU(2), which consists of the three elements T_1, T_2, T_3 obeying $[T_1, T_2] = T_3$, $[T_3, T_1] = T_2$ and $[T_2, T_3] = T_1$. Let us take T_3 to be the generator of a U(1) subalgebra and define M to be the span of T_1 and T_2. The above conditions are then clearly satisfied. So SU(2)/U(1) is a symmetric space.
b) The Lie algebra of SU(n) consists of traceless antihermitian matrices, i.e. matrices of the form

$$\begin{pmatrix} ia_{11} & a_{12}+ib_{12} & \dots & a_{1n}+ib_{1n} \\ -a_{12}+ib_{12} & ia_{22} & \dots & \vdots \\ \vdots & & \ddots & \\ & & \dots & -i(a_{11}+\dots+a_{n-1,n-1}) \end{pmatrix} \tag{9.2}$$

where the as and bs are all real. So simply by separating this into a sum of real and imaginary parts we get a subspace of real matrices which are antisymmetric - and so give the Lie algebra of SO(n) - and a subspace of imaginary symmetric traceless matrices. To see that the latter don't form a Lie algebra let A, B be matrices of this type, and consider the commutator $[A, B]$. This must be real, and therefore this subspace is not closed under the Lie bracket and so is not a Lie algebra.

We now identify $H = \mathrm{SO}(n)$ and M with the subspace of imaginary symmetric traceless matrices. We note that if A, B are imaginary symmetric traceless then $[A, B]$ is real, as we noted above, and also antisymmetric, and so the commutator lies in $L(\mathrm{SO}(n))$. This provides the condition $[M, M] \subset L(\mathrm{SO}(n))$. Also, if we take A to be in $L(\mathrm{SO}(n))$ and B to be imaginary, traceless and symmetric then $[A, B]$ will be imaginary, traceless and symmetric: $[A, B]^T = B^T A^T - A^T B^T = -BA + AB = [A, B]$. So we find that $[L(\mathrm{SO}(n)), M] \subset M$. This, along with the fact $L(\mathrm{SO}(n))$ is a Lie algebra, shows that SU(n)/SO(n) is a symmetric space.

c) If we multiply all the imaginary matrices by i, then the direct sum of the two subspaces turns out to consist of arbitrary real traceless matrices, which is the Lie algebra of SL(n;$\mathbb{R}$). The brackets have changed by various factors of i, but we still find that the conditions hold for SL(n;$\mathbb{R}$)/SO(n) to be a symmetric space.

Question 10

We consider the real Lie algebra with the following brackets

$$[X_i, X_j] = c_{ijk} X_k \quad [X_i, Y_j] = c_{ijk} Y_k \quad [Y_i, Y_j] = -c_{ijk} X_k. \tag{10.1}$$

We know that c_{ijk} are the structure constants for a simple Lie algebra of compact type, with Killing form $\kappa_{ij} = c_{ikl} c_{jlk} = -\delta_{ij}$. We will now work out the Killing form of the above Lie algebra, using barred indices to denote indices associated to the generators Y_i, so that the structure constants are

$$f_{ijk} = c_{ijk} \quad f_{i\bar{j}\bar{k}} = c_{ijk} \quad f_{\bar{i}\bar{j}k} = -c_{ijk}. \tag{10.2}$$

Then we have

$$\kappa_{ij} = f_{ikl} f_{jlk} + f_{i\bar{k}\bar{l}} f_{j\bar{l}\bar{k}} + f_{i\bar{k}l} f_{jl\bar{k}} + f_{ik\bar{l}} f_{j\bar{l}k} = 2 c_{ikl} c_{jlk} = -2\delta_{ij}, \tag{10.3}$$

and similarly one finds

$$\kappa_{i\bar{j}} = 0 \quad \kappa_{\bar{i}\bar{j}} = +2\delta_{ij} \tag{10.4}$$

so that

$$\kappa = \begin{pmatrix} -2I & 0 \\ 0 & +2I \end{pmatrix} \tag{10.5}$$

which is of split signature and so not negative definite.

Question 11

A choice for the Weyl basis[1] is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{11.1}$$

We want to compute the representation of the Lorentz algebra defined by

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu] = \frac{1}{2}\gamma^\mu\gamma^\nu, \mu \neq \nu. \tag{11.2}$$

This gives simply

$$S^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{11.3}$$

and

$$\begin{aligned} S^{ij} &= \frac{1}{2}\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix}, i \neq j \\ &= -\frac{i}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \end{aligned} \tag{11.4}$$

These matrices decompose into block diagonal pieces and so the representation defined by $S^{\mu\nu}$ is reducible. Defining $K_i = S^{0i}$ and $J_i = \frac{1}{2}\varepsilon_{ijk}S^{jk}$ it's easy to see that in the first reducible piece we have the representation

$$K_i = -\frac{1}{2}\sigma_i \quad , \quad J_i = -\frac{i}{2}\sigma^i \tag{11.5}$$

while in the second we have

$$K_i = +\frac{1}{2}\sigma_i \quad , \quad J_i = -\frac{i}{2}\sigma^i \tag{11.6}$$

so that the boost generators are represented by matrices of opposite signs while the rotation generators are the same.

To precisely identify these representations, let's recall that the Lorentz Lie algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho} \tag{11.7}$$

implies that for $J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}$ and $K_i = M^{0i}$ the commutators

$$[J_i, J_j] = \varepsilon_{ijk}J_k \quad [J_i, K_j] = \varepsilon_{ijk}K_k \quad [K_i, K_j] = -\varepsilon_{ijk}J_k. \tag{11.8}$$

We can define instead

$$N_i = \frac{1}{2}(J_i + iK_i) \quad \overline{N}_i = (J_i - iK_i) \tag{11.9}$$

which satisfy

$$[N_i, N_j] = \varepsilon_{ijk}N_k \quad [\overline{N}_i, \overline{N}_j] = \varepsilon_{ijk}\overline{N}_k \quad [N_i, \overline{N}_j] = 0 \tag{11.10}$$

i.e. we get two commuting versions of the SU(2) Lie algebra. We use these commuting SU(2)s to label representations of the Lorentz group as (n,m) with n the highest weight of the representation of the N_i algebra and m the highest weight of the representation of the $\overline{N}_i$ algebra.

In particular for the representation (11.5) we have

$$N_i = -\frac{i}{2}\sigma^i \quad \overline{N}_i = 0 \tag{11.11}$$

which gives the $(\frac{1}{2},0)$ representation, while for the representation (11.6) we instead have

$$N_i = 0 \quad \overline{N}_i = -\frac{i}{2}\sigma^i \tag{11.12}$$

which gives the $(0,\frac{1}{2})$ representation. So a Dirac spinor in four dimensions transforms in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz algebra.

A physical argument to show that $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ is not equivalent to the $(\frac{1}{2},\frac{1}{2})$ representation is the following. Note that in a given representation of the algebras N_i and $\overline{N}_i$ the rotation generators are recovered by $J_i = N_i + \overline{N}_i$. To find the corresponding weights of the representation of the J_i we use the usual method of addition of angular momentum (equivalently, the tensor product decomposition of two representations of the SU(2) Lie algebra). In $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ we clearly can only get spin 1/2 representations of the rotation generators. However in $(\frac{1}{2},\frac{1}{2})$ we will get both a spin one and a spin zero representation (from $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$). So this must be an inequivalent representation to $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$.

[1] I unfortunately don't know exactly what conventions were used in the lectures so in this question I am following those used in David Tong's QFT notes (mostly minus metric and so on).